cat

0556

Tech. Info. Section

AD - 402 391 L

Report No. 7824

LL 73

Copy No. ________

58

AD402 391

20030617125

Land Locomotion Laboratory

Components Research and Development Laboratories

Research and Engineering Directorate

ON THE STATISTICAL ANALYSIS OF LINEAR VEHICLE DYNAMICS

By

J. L. Bogdanoff

F. Kozin

March 1962

PROJECT NO. 5510.11.822

Authenticated: [signature]

D/A Project: 570-05-001

Approved: S H Fuller

U. S. ARMY TANK-AUTOMOTIVE CENTER

Detroit Arsenal

Center Line, Michigan

~~20020814009~~

[redacted]

AD29590

"THE FINDINGS IN THIS REPORT ARE NOT TO BE CONSTRUED AS AN OFFICIAL DEPARTMENT OF THE ARMY POSITION"

LL 73

Linear vehicle dynamics
Statistical analysis
Rough Track
Ride roughness

TABLE OF CONTENTS

ABSTRACT

Statistical analyses of the dynamics of some two-dimensional linear vehicles traveling on a rough track are performed to determine the influence on two aspects of vehicle ride of a set of parameters which include wheel base length, idealized tire imprint length, speed, and damping constant. It is assumed that the vehicles move with constant horizontal velocity on a second order, weakly stationary and mean square continuous random track with contact maintained at all times between the idealized tires and the track. The two aspects of vehicle ride used as measures of the ride roughness are peak value of power spectral density and variance of frame acceleration, the frame acceleration being either vertical at the c. g. of frame, vertical at the point over idealized wheel, or angular (pitching).

For the same speed, damping, power spectral density for the track, and two particular vehicles, the idealized tire imprint length was a relatively unimportant parameter over a fairly large range of values. On the other hand, one parameter which included the wheel base length was found to be important under the same conditions.

Four sets of parameter values were found which at the same speed produced best or optimal rides for vertical acceleration at the frame c. g. and over the wheel, depending upon which measure of ride roughness was employed. The influence of speed was then examined on vehicles having these sets of parameter values. In all cases, increasing speeds produced sharp increases in ride roughness.

PREFACE

This is the sixth in a series of papers studying the motion of vehicles under random excitation. Prior papers are:

a. Land Locomotion Report No. 48, Behavior of a Linear One Degree of Freedom Vehicle Moving With Constant Velocity on a Stationary Gaussian Random Track.

b. Land Locomotion Report No. 56, On the Behavior of a Linear Two Degree of Freedom Vehicle Moving With Constant Velocity on a Track Whose Contour is a Stationary Random Process.

c. Land Locomotion Report No. 65, On the Statistical Analysis of the Motion of Some Simple Vehicles Moving on a Random Track, which also includes an appendix which defines many terms and derives many basic concepts used in the series.

d. Land Locomotion Report No. 66, On the Statistical Analysis of the Motion of Some Simple Two-Dimensional Linear Vehicles Moving on a Random Track.

e. Land Locomotion Report No. 72, On the Statistical Properties of the Ground Contour and its Relation to the Study of Land Locomotion.

Acknowledgement

The authors wish to express their thanks to M. G. Bekker, and Capt. Liston and Capt. Tyler of the Land Locomotion Laboratory, Detroit Arsenal, for helpful discussion given during the course of the investigation.

LIST OF FIGURES

I. Introduction

The motivation for the investigation reported in this paper and the need for a statistical description of the ground in the investigation are described in the Introduction to the preceding paper. Hence, the remarks of this section only need be directed to those aspects of vehicle dynamics and "optimal ride" not previously commented upon.

A wide variety of one, two, and three dimensional models have been suggested for a study of the dynamics of real vehicles. Some of these models are linear while others are non-linear. The one we shall employ in this paper [1] [2] is sufficiently simple so that the mathematics is reasonably tractable and yet sufficiently complicated so as to make parameter studies interesting. A sketch of this linear two-dimensional vehicle is given in Figure 1.

This vehicle consists of a rigid frame connected at each end via a parallel arranged spring and damper suspension element to an idealized wheel-tire. In the initial stages of the investigation, a number of wheel-tire models were considered. However, on simplifying each of them to a case which could be described in a reasonable mathematical form, the wheel-tire model shown was obtained. It has mass but no moment of inertia, and it is connected to the ground or track through a spring-damper arrangement distributed along a massless rigid bar. Contact with the track over the entire foot-pring length is assumed. Thus, the idealized tire acts much like a deformable slipper gliding over the track. Bouncing is not permitted in the sense that contact over the entire foot-print length is maintained at all times, but the wheel mass may oscillate vertically on the suspension element and the idealized tire. We also assume that the elements shown vertical always remain in that position. The vehicle of Figure 1 thus possesses the frame translation and pitch and some of the simpler aspects of

wheel-tire dynamics common to a real vehicle. However, it is clear that frame roll, tire bounding off the track, etc. are excluded.

To estimate whether one vehicle design can be driven faster than another design, or whether a given vehicle design can be driven at a definite speed over a terrain of specified roughness requires that there be some quantitative ride criterion against which projected vehicle capability can be measured. At the present time, a universally accepted ride criterion is not available.

It is a commonly held view among individuals associated with the military wheeled-vehicle field that such vehicles have the reserve power and sufficient mechanical strength to go to much greater speeds than presently being attained under off-road conditions. Moreover, if speeds are reached that begin to break parts, or if more power is needed to increase speed, these individuals believe that the needed strength and power can easily be supplied. The driver, occupants, or cargo of vehicles thus appears to provide a basic limitation on vehicle speed.

The frame motion of the vehicle model is limited to vertical translation and pitch. The acceleration of each appears to influence certain aspects of ride. For example, bouncing of the driver, say, upon his seat is associated with large vertical frame acceleration. When such bouncing becomes sufficiently severe, it results in physical injury. Severe oscillatory acceleration in pitch results in loss of visual perception (this also occurs with severe oscillatory vertical acceleration) and loss of equilibrium. In most cases, the driver reduces vehicle speed before either the oscillating vertical or pitching acceleration reaches such extreme values, or the occupants insist that he do so. Damage tests with humans to determine when vertical acceleration causes injury, pitch acceleration causes loss of visual perception, etc., have been conducted using simple harmonic motions. [3] [4] Results of such tests cannot be directly converted to the case in which the oscil-

lations are random in character. Hence, bounds on vertical acceleration (also transverse, etc.) and pitching acceleration set by these two tests are not useful to us. Tests in progress and also projected will be with randomly oscillating acceleration, and, when these results are available, they should establish quantitative bounds on human tolerance to acceleration in random oscillation. In any event, it is reasonable at this time to base ride criteria on random oscillatory acceleration in vertical acceleration and pitch.

A number of drivers of military vehicles also pointed out that when large gun tubes are carried in a turret, limiting vehicle speed was frequently set by violent oscillations occuring in these tubes as a result of severe vehicle pitch.

We shall select two quantities as measures of the severity of a random oscillation. One is the magnitude of the largest peak in the power spectral density. The smaller the maximum peak, the lower the ordinate bounding the entire spectral density curve. While the work of Goldman [3] is not applicable, it does suggest that low values of this bounding ordinate may be associated with a comfortable ride from the human standpoint in random oscillatory acceleration.

The variance is a measure of the spread of a probability distribution about its mean. A large variance implies that the probability of a significant deviation from the mean is large. For a reasonably smooth random oscillation, the variance may thus be used as a measure at any time of the probability of the oscillation deviating from its mean by some specified amount. Hence, by reducing variance, the probability of large excursion in such an oscillation is in general reduced.

It is well to point out that other aspects of vehicle acceleration may provide useful ride criteria. In particular, the work of von Gierke [4] and others [5] suggests that minimum acceleration

power spectral density (vertical translation or pitch) in a band from 3 c p s to 10 c p s might produce the best ride from the human standpoint. Thus, the peak in this band could be a measure of the severity of a random acceleration.

In this paper, we shall study the influence of several vehicle parameters upon an optimal ride criterion based either on minimizing the largest peak in power spectral density or on minimizing the variance in vertical translation or pitch acceleration. All numerical results presented pertain to these optimal ride criteria.

We shall begin the main portions of the paper by presenting some elementary remarks on such statistical quantities as mean, variance, covariance, power spectral density, m. s. continuity, etc. as they apply to the track on which the model moves. The physical aspects of the model considered in the equations of motion are then discussed. Section IV presents the equations of motion and shows how the power spectral density of vertical translation of the frame, etc. are obtained. Numerical results on parameter studies for two special cases of the vehicle of Figure 1 and some general observations are given in the final section.

II. Track Model

A detailed discussion of a stochastic model of ground roughness is being presented elsewhere [2] . It suffices for the purposes of this paper to simply assume that the track elevation vs horizontal distance is a random function possessing statistical properties that ground roughness along a line of reasonable length is likely to possess. In particular, we shall confine our assumptions to second order properties [7] , i. e. to properties which can be deduced from covariances. Thus, explicit probability distributions of ground roughness or elevation are not employed.

The mean response of a linear vehicle to a non-zero mean track elevation may be treated by deterministic and well known methods. The fluctuations of the response about the mean motion, on the other hand, must be treated by probabilistic methods. Since we are primarily interested in the latter, we shall assume that the mean track elevation has the constant value zero. As we wish to employ second order properties, it is required that the track of elevation be a second order [7] random function (r. f.). Member functions of ground elevations are, of course, oscillatory in nature, but they should not contain discontinuities involving more than finite jumps. The assumption of mean square (m. s.) continuity [7] is usually sufficient to eliminate unreasonable member functions and we shall employ it. Finally, in order to be able to employ the concept of power spectral density and the techniques associated with it [8],we shall assume that ground elevation vs horizontal distance is at least weakly stationary [7] . This means that at least the first two probability distributions do not depend upon the location of the origin of the horizontal distance.

As an additional comment upon the fact that results given in this paper only depend upon second order properties and not upon particular types of distributions, let us note that whether a particular representation of ground elevation as a r. f. is Gaussian or not is of no consequence.

There are essentially two ways in which it is convenient to represent ground or track elevation $Y_0(x)$ as a r. f. of the horizontal coordinate x. In one way, an explicit formula for $Y_0(x)$ in terms of a family or set of random variables (r. v's.) is adopted, and, in the other, certain statistical requirements are postulated (zero mean, second order, m. s. continutiy, and at least weak stationarity) and a certain integral representation meeting these requirements employed. Each procedure has certain advantages; hence we shall discuss them.

A formula adopted in previous papers [1] [2] is

$$Y_0(x) = 2\sum_{j=1}^{n} a_j \cos(\lambda_j x + \Phi_j) \tag{2.1}$$

where a_j and λ_j are ordinary constants and the Φ_j are independent r. v's. uniformly distributed over an interval of length 2π Thus, $Y_0(x)$ is a sum of cosine functions having randomly distributed phases. The λ_j have the dimensions of length to the minus one power; and, when divided by 2π give cycles per unit length. Hence, they may be regarded as angular "length" frequencies. The a_j are, of course, the amplitudes of the simple harmonic elements. The statistical properties of (2. 1) are well known [9]; we easily find that

$$(2.2)\quad \begin{cases} E\left\{Y_0(x)\right\} = 0 \\ \Gamma_{Y_0}(x_1, x_2) = E\left\{Y_0(x_1)\; Y_0(x_2)\right\} = 2\sum_{j=1}^{n} a_j^2 \cos \lambda_j (x_2 - x_1) \\ \sigma_{Y_0}^2(x) \quad = \Gamma_{Y_0}(x, x) = 2\sum_{j=1}^{n} a_j^2 \end{cases}$$

It is perhaps appropriate to review briefly the meanings of these equations. The first states that the mean of $Y_0(x)$ is zero. The third states that $\sigma^2_{Y_0}(x)$ is independent of x and since for finite a_j and finite n it is finite, $Y_0(x)$ is of second order. $\Gamma_{Y_0}(x_1, x_2)$ is differentiable any number of times with respect to x_1 and x_2; thus, $Y_0(x)$ is m. s. differentiable any number of times. (Indeed, the member functions are differentiable to any order.) Finally, since $\Gamma_{Y_0}(x_1, x_2)$ depends only upon $x_2 - x_1$ and the mean is constant, we know that $Y_0(x)$ is at least weakly stationary. Hence, the requirements enumerated above for the random function which is to represent ground elevation vs horizontal are met.

If n is replaced by infinity, there are conditions--which we need not state here--on the a_j and λ_j if $Y_0(x)$ is to be second order and m. s. continuous.

The power spectral density of $Y_0(x)$ is given by the formula [9]

$$(2.3) \qquad P_{Y_0}(\lambda) = 2\sum_{j=1}^{n} a_j^2\, \delta(\lambda - \lambda_j)$$

where $\delta(\lambda - \lambda_j)$ is the delta function centered at λ_j with the properties

$$\int_{-\infty}^{\infty} \delta(\lambda - \lambda_j)\, d\lambda = 1 \, , \quad \delta(\lambda - \lambda_j) = 0 \, , \text{ if } \lambda \neq \lambda_j$$

We shall call $2a_j^2$ the weight of the power spectral density at λ_j. If $Y_0(x)$ is interpreted as a voltage applied to a resistance of one ohm and x is interpreted as the time, then the integral of $P_{Y_0}(\lambda)d\lambda$ between λ_a and $\lambda_b (> \lambda_a)$ is the average power in $Y_0(x)$ between

those values of λ . On setting $x_2 - x_1 = x_0$ we may write down the well known [8] formulas

$$(2.4) \qquad \begin{cases} \Gamma_{Y_0}(x_0) = \int_0^\infty P_{Y_0}(\lambda) \cos \lambda x_0 \, d\lambda \\ P_{Y_0}(\lambda) = \frac{2}{\pi} \int_0^\infty \Gamma_{Y_0}(x_0) \cos \lambda x_0 \, dx_0 \\ \sigma_{Y_0}^2 = \int_0^\infty P_{Y_0}(\lambda) d\lambda \end{cases}$$

That is, the covariance $\Gamma_{Y_0}(x_0)$ and the power spectral density $P_{Y_0}(\lambda)$ are Fourier transforms of one another.

If a set $\phi_1, \ldots, \phi_n$ of the Φ_j are selected with the aid of a table of random numbers, say, then one member function

$$(2.5) \qquad y_0(x) = 2 \sum_{j=1}^{n} a_j \cos(\lambda_j x + \phi_j)$$

of $Y_0(x)$ is obtained. Different selections of the ϕ_j lead to different member functions. We have shown elsewhere [1] how it is possible to select the a_j and λ_j so that the integrated power spectral density

$$\int_0^\lambda P_{Y_0}(\lambda') d\lambda'$$

of (2.3) approximates as closely as is required to the integral

$$\int_0^\lambda P(\lambda') d\lambda'$$

of some prescribed or given power spectral density $P(\lambda)$. Since such a procedure determines the a_j and λ_j , it is thus possible with (2.5)

to obtain a deterministic member function $y_0(x)$ of the random function (2.1), and the integrated power spectral density of this $Y_0(x)$ approximates to the integral of the prescribed power spectral density $P(\lambda)$. A graph, or for that matter, an actual track* of a section of the member function (2.5) belonging to this $Y_0(x)$ can therefore be constructed.

The advantages in employing or considering (2.1) are that a simple and readily interpretable formula for $Y_0(x)$ is available, member functions are easy to obtain and present in graphical form, if desired, and the power spectral density (2.3) possesses an interpretation in terms of well known electrical engineering concepts. These advantages of (2.1) prompted us to introduce it for the benefit of those readers whose background in random function theory needs refreshing. The disadvantage in using (2.1) in analytical work stems from the fact it often leads to a good deal of unnecessary algebra.

Let us now examine the formula

$$(2.6) \qquad Y_0(x) = \int_{-\infty}^{\infty} e^{i\lambda x}\, dZ(\lambda)$$

where the integral is to be interpreted in the m. s. sense [7] and the random set function $Z(\lambda)$ is an orthogonal process [7] with the properties

---

* A track calculated on this basis has been constructed at the Detroit Arsenal.

$$
(2.7)\quad \begin{cases}
E\left\{Z\right\} = 0 \\
\dfrac{d}{d\lambda} E \left\{ |Z(\lambda)|^2 \right\} = \dfrac{1}{2} P_{Y_0}(\lambda) \\
P_{Y_0}(\lambda) = P_{Y_0}(-\lambda) \\
\dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} P_{Y_0}(\lambda)\, d\lambda < \infty, \quad \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} \lambda^2 P_{Y_0}(\lambda)\, d\lambda < \infty
\end{cases}
$$

From these equations, we find that

$$
(2.8)\quad \begin{cases}
E\left\{Y_0(x)\right\} = 0 \\
\Gamma_{Y_0}(x_1, x_2) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} e^{i\lambda(x_2 - x_1)} P_{Y_0}(\lambda)\, d\lambda \\
\qquad\qquad = \displaystyle\int_{0}^{\infty} P_{Y_0}(\lambda) \cos \lambda x_0 \, d\lambda \\
\sigma_{Y_0}^2(x) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} P_{Y_0}(\lambda)\, d\lambda < \infty
\end{cases}
$$

Thus, $Y_0(x)$ has a zero mean, is of second order, and, since the covariance depends only upon $x_2 - x_1$ and the mean is constant, it is at least weakly stationary. The last of (2.7) and second of (2.8) enable us to demonstrate the m. s. continuity of (2.6). Hence, all the requirements which we wish to impose on the r. f. representing ground or track elevation as horizontal distance are met.

In what follows, we shall employ (2.6) rather than (2.1), because the former lends itself to certain computations we wish to make.

III. Vehicle Model

Figure 1 shows the idealized linear two-dimensional vehicle that is to be discussed. The rigid frame has wheel base length l ; its mass is M; and its moment of inertia about its c. g. is I_m. The c. g. is at distance b from the wheel base center O. $Y(t)$ is the vertical displacement of O, and $\theta(t)$ is the angular displacement of the frame. We shall assume that O has a constant horizontal speed v so that its x-coordinate is vt, t being the time.

The ends of the frame are supported on linear springs and viscous dampers arranged in parallel. k_j and c_j $(j=1,2)$ are the rear $(j=1)$ and front $(j=2)$ spring and viscous damper constants respectively. These elements are taken as massless, the upper and lower pivots are assumed frictionless, and they remain vertical throughout the motion.

Idealized tires are assumed. Each has a rigid massless frame of length $2a_j$ and a mass m_j concentrated at the frame center. These frames are connected by frictionless pivots to the bottoms of the parallel arranged spring and damper elements, as shown. $K_j(\xi_j)$ and $\gamma_j(\xi_j)$ are, respectively, the linear spring constant per unit length and the viscous damping constant per unit length which are distributed in a parallel arrangement over the bottoms of the frames; we assume these elements are massless. We shall make the initial assumption that $K_j(\xi_j)$ and $\gamma_j(\xi_j)$ are even functions of ξ_j ; at a latter stage, we shall simply assume they are independent of ξ_j. Contact of the tire spring and damper elements with the track a road is assumed to be maintained at all times. $\theta_1(t)$ is the angular displacement of the rear tire frame and $\theta_2(t)$ is the angular dispplacement of the front frame. The vertical displacements of the masses m_1 and m_2 are $Y_1(t)$ and $Y_2(t)$, respectively.

Gravity will be neglected. Further, all angles are assumed to be sufficiently small so that cosines may be replaced by unity and the sines by the angles in radians.

IV. Analysis of Vehicle Dynamics

For the vehicle model just described, we find the equations of motion to be

$$(4.1)\quad \begin{cases} M(\ddot{Y}+b\ddot{\Theta}) = -F_1(t) - F_2(t), \\ I_m\ddot{\Theta} = (\frac{1}{2}+b)F_1(t) - (\frac{1}{2}-b)F_2(t), \\ m_1\ddot{Y}_1 = F_1(t) - F_3(t), \\ m_2\ddot{Y}_2 = F_2(t) - F_4(t), \end{cases}$$

where

$$(4.2)\quad \begin{cases} F_1(t) = k_1(Y - \frac{1}{2}\Theta - Y_1) + c_1(\dot{Y} - \frac{1}{2}\dot{\Theta} - \dot{Y}_1), \\ F_2(t) = k_2(Y + \frac{1}{2}\Theta - Y_2) + c_2(\dot{Y} + \frac{1}{2}\dot{\Theta} - \dot{Y}_2), \\ F_3(t) = \int_{-a_1}^{a_1} \{K_1(\xi_1)[Y_1 + \xi_1 \Theta_1 - Y_0(x - \frac{1}{2} + \xi_1)] \\ \qquad + \gamma_1(\xi_1)[\dot{Y}_1 + \xi_1\dot{\Theta}_1 - \dot{Y}_0(x - \frac{1}{2} + \xi_1)]\}\, d\xi_1, \\ F_4(t) = \int_{-a_2}^{a_2} K_2(\xi_2)[Y_2 + \xi_2\Theta_2 - Y_0(x + \frac{1}{2} + \xi_2)] \\ \qquad + \gamma_2(\xi_2)[\dot{Y}_2 + \xi_2\dot{\Theta}_2 - \dot{Y}_0(x + \frac{1}{2} + \xi_2)]\}\, d\xi_2 \end{cases}$$

and where derivatives and integrals are to be interpreted in the m. s. sense [7].

To simplify the analysis from this point on, it is convenient to assume that the vehicle is symmetrical, i. e.

$$
(4.3)\quad \begin{cases}
k_1 = k_2 = k \quad , \quad c_1 = c_2 = c \; , \\
a_1 = a_2 = a \quad , \quad K_1(\xi) = K_2(\xi) = K(\xi) \; , \\
\gamma_1(\xi) = \gamma_2(\xi) = \gamma(\xi) \; , \quad m_1 = m_2 = m \; , \\
b = 0
\end{cases}
$$

The equations of motion may now be written in the form

$$
(4.4)\quad \begin{cases}
\ddot{Y} + 2\zeta_y \omega_y \dot{Y} + \omega_y Y - \frac{1}{2}(2\zeta_y \omega_y \dot{Y}_1 + \omega_y^2 Y_1) - \frac{1}{2}(2\zeta_y \omega_y \dot{Y}_2 + \omega_y^2 Y_2) = 0 \; , \\
\ddot{\Theta} + 2\zeta_\theta \omega_\theta \dot{\Theta} + \omega_\theta \Theta + \frac{1}{l}(2\zeta_\theta \omega_\theta \dot{Y}_1 + \omega_\theta^2 Y_1) - \frac{1}{l}(2\zeta_\theta \omega_\theta \dot{Y}_2 + \omega_\theta^2 Y_2) = 0 \; , \\
-\alpha_1 \left(\frac{2\zeta_y \omega_o^2}{\omega_y} \dot{Y} + \omega_o^2 Y\right) + \frac{\alpha_1 l}{2} \left(\frac{2\zeta_\theta \omega_o^2}{\omega_\theta} \dot{\Theta} + \omega_o^2 \Theta\right) \\
+ \ddot{Y}_1 + \left(2\zeta \omega_o + \alpha_1 \frac{2\zeta_y \omega_o^2}{\omega_y}\right) \dot{Y}_1 + \omega_o^2 (1 + \alpha_1) Y_1 \\
= \omega_o^2 \int_{-a}^{a} \left[\frac{K(\xi)}{K} Y_o\left(x - \frac{l}{2} + \xi\right) + \frac{2\zeta_o}{\omega_o} \frac{\gamma(\xi)}{C} \dot{Y}_o\left(x - \frac{l}{2} + \xi\right)\right] d\xi \; , \\
-\alpha_1 \left(\frac{2\zeta_y \omega_o^2}{\omega_y} \dot{Y} + \omega_o^2 Y\right) - \frac{\alpha_1 l}{2} \left(\frac{2\zeta_\theta \omega_o^2}{\omega_\theta} \dot{\Theta} + \omega_o^2 \Theta\right) \\
+ \ddot{Y}_2 + \left(2\zeta_o \omega_o + \alpha_1 \frac{2\zeta_y \omega_o^2}{\omega_y}\right) \dot{Y}_2 + \omega_o^2 (1 + \alpha_1) Y_2 \\
= \omega_o^2 \int_{-a}^{a} \left[\frac{K(\xi)}{K} Y_o\left(x + \frac{l}{2} + \xi\right) + \frac{2\zeta_o}{\omega_o} \frac{\gamma(\xi)}{C} \dot{Y}_o\left(x + \frac{l}{2} + \xi\right)\right] d\xi
\end{cases}
$$

where now

$$
(4.5) \quad \begin{cases} \omega_y^2 = \frac{2k}{M} \ , \quad 2\zeta_y \omega_y = \frac{2c}{M} \ , \quad \omega_\theta^2 = \frac{k l^2}{2 I_m} \ , \\ 2\zeta_\theta \omega_\theta = \frac{c l^2}{2 I_m} \ , \quad \omega_0^2 = \frac{K}{m} \ , \quad 2\zeta_0 \omega_0 = \frac{C}{m} \ , \\ \alpha_1 = \frac{k}{K} \ , \quad K = \int_{-a}^{a} \kappa(\xi) d\xi \ , \quad C = \int_{-a}^{a} \gamma(\xi) d\xi \end{cases}
$$

These symbols have the usual interpretations given in small vibration theory. The angles $\theta_1(t)$ and $\theta_2(t)$ drop out of the third and fourth of (4. 4) since we have taken $\kappa(\xi)$ and $\gamma(\xi)$ to be even functions of ξ.

We must now employ a representation for $Y_0(x)$. As mentioned above, it is convenient to use (2. 6). It is also convenient at this time to assume the forms of $\kappa(\xi)$ and $\gamma(\xi)$ are the same. Thus,

$$
\frac{\kappa(\xi)}{K} = \frac{\gamma(\xi)}{C} = \frac{g(\xi)}{2a} \ ,
$$

where $g(\xi)$ is even in ξ and the two non zero terms on the right hand sides of (4. 4) become

$$
(4.6) \quad \omega_0^2 \int_{-\infty}^{\infty} e^{\pm \frac{i\omega l}{2V}} G\left(\frac{\omega a}{V}\right)\left(1 + i\,\frac{2\zeta_0 \omega}{\omega_0}\right) e^{i\omega t}\, dZ\left(\frac{\omega}{V}\right)
$$

respectively, where

$$
(4.7) \qquad \omega = \lambda v \ , \quad G\left(\frac{\omega a}{v}\right) = \frac{1}{2a} \int_{-a}^{a} g(\xi) e^{\frac{i\omega \xi}{v}}\, d\xi
$$

With v a constant, $Y_0(vt)$ and $\dot{Y}_0(vt)$ are still at least weakly stationary. It follows [8] that $Y(t)$, $\Theta(t)$, $Y_1(t)$, $Y_2(t)$ are also at least weakly stationary, since they are related to $Y_0(vt)$ and its m. s. derivative $\dot{Y}_0(vt)$ through a system of constant coefficient linear differential equations. Thus, we may write

$$
(4.8) \qquad \begin{cases} Y(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dU(\omega) \\ \Theta(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dV(\omega) \\ Y_1(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dW_1(\omega) \\ Y_2(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dW_2(\omega) \end{cases}
$$

the random set functions $U(\omega)$, $V(\omega)$, $W_1(\omega)$, $W_2(\omega)$ being orthogonal processes. The substitution of (4.6) and (4.8) into (4.4) provides us with the following equations for the determination of $dU(\omega)$, $dV(\omega)$, $dW_1(\omega)$ and $dW_2(\omega)$:

$$
(4.9)\quad \left\{
\begin{aligned}
&(\omega_y^2-\omega^2+i2\zeta_y\,\omega_y\omega)\,dU(\omega)-\frac{1}{2}(\omega_y^2+i2\zeta_y\,\omega_y\,\omega)\,dW_1(\omega)\\
&-\frac{1}{2}(\omega_y^2+i2\zeta_y\,\omega_y\,\omega)\,dW_2(\omega)=0\ ,\\
&(\omega_\theta^2-\omega^2+i2\zeta_\theta\,\omega_\theta\omega)\,dV(\omega)+\frac{1}{l}(\omega_\theta^2+i2\zeta_\theta\,\omega_\theta\,\omega)\,dW_1(\omega)\\
&-\frac{1}{l}(\omega_\theta^2+i2\zeta_\theta\,\omega_\theta\omega)\,dW_2(\omega)=0\ ,\\
&-\alpha_1\left(\omega_0^2+i\,\frac{2\zeta_y\omega_0^2\omega}{\omega_y}\right)dU(\omega)+\frac{\alpha_1 l}{2}\left(\omega_0^2+i\,\frac{2\zeta_\theta\,\omega_0^2\omega}{\omega_\theta}\right)dV(\omega)\\
&+\left[\omega_0^2\,(1+\alpha_1)-\omega^2+i\left(2\zeta_0\,\omega_0\omega+\alpha_1\,\frac{2\zeta_y\,\omega_0\omega}{\omega_y}\right)\right]dW_1(\omega)\\
&=\omega_0^2\,e^{-\frac{i\omega l}{2v}}\,G\left(\frac{\omega a}{v}\right)\left(1+i\,\frac{2\zeta_0\,\omega}{\omega_0}\right)dZ\left(\frac{\omega}{v}\right)\ ,\\
&-\alpha_1\left(\omega_0^2+i\,\frac{2\zeta_y\,\omega_0^2\omega}{\omega_y}\right)dU(\omega)-\frac{\alpha_1 l}{a}\left(\omega_0^2+i\,\frac{2\zeta_\theta\,\omega_0^2\omega}{\omega_\theta}\right)dV(\omega)\\
&+\left[\omega_0^2\,(1-\alpha_1)-\omega^2+i\left(2\zeta_0\,\omega_0\omega+\alpha_1\,\frac{2\zeta_y\,\omega_0^2\omega}{\omega_y}\right)\right]dW_2(\omega)\\
&=\omega_0^2\,e^{\frac{i\omega l}{2v}}\,G\left(\frac{\omega a}{v}\right)\left(1+i\,\frac{2\zeta_0\,\omega}{\omega_0}\right)dZ\left(\frac{\omega}{v}\right)
\end{aligned}
\right.
$$

The solution of the first two of (4. 9) for $dW_1(\omega)$ and $dW_2(\omega)$ gives

$$
(4.10)\qquad \begin{cases} dW_1(\omega) = \dfrac{-w_1 dU(\omega) + w_2 dV(\omega)}{w} , \\[2ex] dW_2(\omega) = -\dfrac{w_1 dU(\omega) + w_2 dV(\omega)}{w} , \end{cases}
$$

with

$$
(4.11)\qquad \begin{cases} w = (\omega_y^2 + i2\zeta_y \omega_y \omega)(\omega_\theta^2 + i2\zeta_\theta \omega_\theta \omega) , \\[1ex] w_1 = 2(\omega_\theta^2 + i2\zeta_\theta \omega_\theta \omega)(\omega_y^2 - \omega^2 + i2\zeta_y \omega_y \omega) \\[1ex] w_2 = l(\omega_y^2 + i2\zeta_y \omega_y \omega)(\omega_\theta^2 - \omega^2 + i2\zeta_\theta \omega_\theta \omega) \end{cases}
$$

The last two of (4. 9) with (4. 10) determine $dU(\omega)$ and $dV(\omega)$:

$$
(4.12)\qquad \begin{cases} dU(\omega) = \dfrac{-z}{u_1 + \frac{u w_1}{w}} \cos \dfrac{\omega l}{2v} \, dZ\left(\dfrac{\omega}{v}\right) , \\[2ex] dV(\omega) = \dfrac{-iz}{v_1 + \frac{u w_2}{w}} \sin \dfrac{\omega l}{2v} \, dZ\left(\dfrac{\omega}{v}\right) , \end{cases}
$$

where

$$
(4.13)\qquad \begin{cases} u_1 = \alpha_1 \left(\omega_0^2 + i \dfrac{2\zeta_y \omega_0^2 \omega}{\omega_y}\right) , \\[2ex] v_1 = \dfrac{\alpha_1 l}{2} \left(\omega_0^2 + i \dfrac{2\zeta \, \omega_0^2 \omega}{\omega_\theta}\right) , \\[2ex] u = \omega_0^2 (1+\alpha_1) - \omega^2 + i\left(2\zeta_0 \omega_0 \omega + \alpha_1 \dfrac{2\zeta_y \omega_0^2 \omega}{\omega_y}\right) , \\[2ex] z = \omega_0^2 \, G\left(\dfrac{\omega a}{v}\right)\left(1 + i \dfrac{2\zeta_0 \omega}{\omega_0}\right) . \end{cases}
$$

The power spectral density of $Y(t)$, for example, is given by a formula similar to the second of (2. 7); it is

$$(4.14) \qquad \frac{1}{2} P_Y(\omega) = \frac{d}{d\omega} E\left\{ |U(\omega)|^2 \right\}$$

Hence, with the first of (4. 12) and the second of (2. 7), we find that

$$(4.15) \qquad P_Y(\omega) = \frac{1}{2v} \left[1 + \left(\frac{2\zeta_0 \omega}{\omega_0} \right)^2 \right] \left[1 + \left(\frac{2\zeta_y \omega}{\omega_y} \right)^2 \right] \left[|\alpha_1 \left(1 + i \frac{2\zeta_y \omega}{\omega_y} \right)^2 \right.$$

$$+ 2 \left\{ 1 + \alpha_1 - \frac{\omega^2}{\omega_0^2} + i2\left(\zeta_0 + \alpha_1 \frac{\zeta_y \omega_0}{\omega_y} \right) \frac{\omega}{\omega_0} \right\} \left(1 - \frac{\omega^2}{\omega_y^2} \right.$$

$$\left. \left. + i \frac{2\zeta_y \omega}{\omega_y} \right) \right|^2 \Big]^{-1} \left(1 + \cos \frac{\omega l}{v} \right) \left[G\left(\frac{\omega a}{v} \right) \right]^2 P_{Y_0}\left(\frac{\omega}{v} \right)$$

and similarly,

$$(4.16) \qquad P_\theta(\omega) = \frac{1}{2vl^2} \left[1 + \left(\frac{2\zeta_0 \omega}{\omega_0} \right)^2 \right] \left[1 + \left(\frac{2\zeta_\theta \omega}{\omega_\theta} \right)^2 \right] \left[| \frac{\alpha_1}{2} \quad 1 + i \frac{2\zeta_\theta \omega}{\omega_\theta} \right.$$

$$+ \left\{ 1 + \alpha_1 - \frac{\omega^2}{\omega_\theta^2} + i2\left(\zeta_0 + \alpha_1 \frac{\zeta_y \omega}{\omega_y} \right) \frac{\omega}{\omega_0} \right\} \left(1 - \frac{\omega^2}{\omega_\theta^2} \right.$$

$$\left. + i \frac{2\zeta_\theta \omega}{\omega_\theta} |^2) \right]^{-1} \left(1 - \cos \frac{\omega l}{v} \right) \left[G\left(\frac{\omega a}{v} \right) \right]^2 P_{Y_0}\left(\frac{\omega}{v} \right)$$

Two special cases of the system shown in Figure 1 are of particular interest in this paper. The first is obtained by replacing the idealized tires by massless point follower. The power spectral densities of the vertical displacement $Y(t) + z\,\theta(t)$ at a point on the frame at a distance z along the frame from the c. g. and of $\theta(t)$ are

$$
(4.17)\quad \left\{
\begin{aligned}
P_Y(z,\omega)_1 = {} & \frac{1}{2l^2}\left[\left(1-\frac{\omega^2}{\omega_y^2}\right)^2+\left(2\zeta_y\frac{\omega}{\omega_y}\right)^2\right]^{-1}\left[\left(1-\frac{\omega^2}{\omega_\theta^2}\right)^2+\left(2\zeta_\theta\frac{\omega}{\omega_\theta}\right)^2\right]^{-1} \\
& \times\left\{l^2\left(1+\cos\frac{\omega l}{v}\right)\left[1+\left(2\zeta_y\frac{\omega}{\omega_y}\right)^2\right]\left[\left(1-\frac{\omega^2}{\omega_\theta^2}\right)^2+\left(2\zeta_\theta\frac{\omega}{\omega_\theta}\right)^2\right]\right. \\
& + 8lz\sin\frac{\omega l}{v}\left[\zeta_y\frac{\omega^3}{\omega_y^3}\left\{\frac{\omega^2}{\omega_\theta^2}-1-\left(2\zeta_\theta\frac{\omega}{\omega_\theta}\right)^2\right\}\right. \\
& \left.-\zeta_\theta\frac{\omega^3}{\omega_\theta^3}\left\{\frac{\omega^2}{\omega_y^2}-1-\left(2\zeta_y\frac{\omega}{\omega_y}\right)^2\right\}\right]+4z^2\left(1-\cos\frac{\omega l}{v}\right)\left[1\right. \\
& \left.\left.+\left(2\zeta_\theta\frac{\omega}{\omega_\theta}\right)^2\right]\left[\left(1-\frac{\omega^2}{\omega_y^2}\right)^2+\left(2\zeta_y\frac{\omega}{\omega_y}\right)^2\right]\right\}\frac{P_{Y_0}\left(\frac{\omega}{v}\right)}{v}, \\[2em]
P_\theta(\omega_4)_1 = {} & \frac{2\left(1\ \cos\frac{\omega l}{v}\right)\left[1+\left(\frac{2\zeta_\theta\omega}{\omega_\theta}\right)^2\right]}{\left[\left(1-\frac{\omega^2}{\omega_\theta^2}\right)^2+\left(\frac{2\zeta_\theta\omega}{\omega_\theta}\right)^2\right]^2 l^2 v}\,P_{Y_0}\left(\frac{\omega}{v}\right),
\end{aligned}
\right.
$$

The power spectral densities of $Y_1(t)$, $Y_2(t)$ and of the vertical motion at points on the vehicle frame other than its c. g. may be found in the same manner.

respectively, where the frequency range is from $-\infty$ to ∞.
The second case is obtained from the system shown in Figure by making the two elements connecting the frame and the tire masses rigid and lumping the tire masses in with the mass of the frame; we find that the spectral densities of $Y(t) + z\,\theta(t)$ and $\theta(t)$ are

$$(4.18)\qquad \begin{cases} P_Y(z,\omega)_2 = \left[G\left(\frac{\omega a}{V}\right)\right]^2 P_Y(z,\omega)_1 \;, \\[2ex] P_\theta(z,\omega)_2 = \left[G\left(\frac{\omega a}{V}\right)\right]^2 P_\theta(z,\omega)_1 \;. \end{cases}$$

We have omitted derivations of (4.17) and (4.18) since they follow almost the same sequence of steps that we employed in obtaining (4.15) and (4.16) from (4.1).

V. Discussion and Results

It is possible to draw a few general conclusions from the results given in the previous section.

Examination of (4. 15) and (4. 16), say, reveals that the half-tire frame a half foot print length a only enters these equations through the term $G(\frac{\omega a}{V})$ where [see (4. 7)],

$$G\left(\frac{\omega a}{V}\right) = \frac{1}{2a}\int_{-a}^{a} g(\xi)\, e^{\frac{i\omega\xi}{V}}\, d\xi$$

and $g(\xi)$ is the distribution along the frame of the parallel arranged linear spring and viscous damper tire elements. Since $G(\frac{\omega a}{V})$ is an integral of $g(\xi)\, e^{\frac{i\omega\xi}{V}}/2a$ over the foot print length, $G(\frac{\omega a}{V})$ represents the smoothing of the roughness in the track due to the idealized tires. Thus,

$$\text{(5.1)} \qquad \left[G\left(\frac{\omega a}{V}\right)\right]^2 P_{Y_0}\left(\frac{\omega}{V}\right)$$

may be regarded as the power spectral density of the input to the vehicle obtained from the power spectral density of $Y_0(x)$ after smoothing by the idealized tires. This suggests that in the analysis we could have ignored the tire-foot print length provided the track $Y_0(x)$ with power spectral density $P_{Y_0}(\frac{\omega}{V})$ were replaced by a smoothed track with power spectral density (5. 1). Advantage of this point may therefore be taken in future investigations. The conclusion that a tire is a smoothing device in some sense is in agreement with results obtained on large low pressure tires.

In the numerical results presented below, we shall assume that

$$g(\xi) = 1, \tag{5.2}$$

and, thus,

$$\left[G\left(\frac{\omega a}{v}\right)\right]^2 = \left(\frac{\sin\frac{\omega a}{v}}{\frac{\omega a}{v}}\right)^2 \tag{5.3}$$

It is clear from this equation that the larger a, the narrower the peak in the term on the right. Hence, the larger a, the smaller the high frequency portion of (5.1) and the greater the smoothing influence of the tires. Conversely, the smaller a, the broader the peak in the right of (5.3), and the less the smoothing effect of the tires. Actually, the selection (5.2) is not particularly realistic, since $g(\xi)$ should approach zero as $\xi \to \pm a$. However, (5.3) is easy to compute with, provides an estimate of the influence of the tire on spectral density of various quantities, and so is adequate for present purposes.

The power spectral density $P_Y(\omega)$ contains the wheel base length l in just one term, namely in $(1+\cos \omega l/v)$. Since this term multiplies $P_{Y_0}(\omega/v)$, just as does $[G(\omega a/v)]^2$, it may also be regarded as a filter acting on $P_{Y_0}(\omega/v)$. Clearly, the term filters periodically, depending upon the value of l/v. Thus, wheel base length and speed v may be expected to have a substantial influence on certain aspects of vehicle ride. The term $(1-\cos \omega l/v)$ in $P_\theta(\omega)$ may be interpreted in the same sense.

From the first of (4.8), we deduce that

$$(5.4) \qquad \Gamma_Y(t_1, t_2) = \frac{1}{2} \int_{-\infty}^{\infty} e^{i\omega(t_2 - t_1)} P_Y(\omega) d\omega .$$

The *r.f.* $Y(t)$ has two m. s. derivates $\dot{Y}(t)$ and $\ddot{Y}(t)$; it follows [8] that the covariances of these r. f. are

$$(5.5) \qquad \begin{cases} \Gamma_{\dot{Y}}(t_1, t_2) = \dfrac{\partial^2}{\partial t_1 \partial t_2} \Gamma_Y(t_1, t_2) , \\ \\ \Gamma_{\ddot{Y}}(t_1, t_2) = \dfrac{\partial^4}{\partial t_1^2 \partial t_2^2} \Gamma_Y(t_1, t_2) . \end{cases}$$

Hence, on combining (5.4) and (5.5) and making use of the definition of the covariances of $\dot{Y}(t)$ and $\ddot{Y}(t)$ in terms of integrals of their power spectral densities, we find that

$$(5.6) \qquad \begin{cases} P_{\dot{Y}}(\omega) = \omega^2 P_Y(\omega) , \\ \\ P_{\ddot{Y}}(\omega) = \omega^4 P_Y(\omega) ; \end{cases}$$

thus, the power spectral density of the first or second m. s. derivative of a suitable r. f. may be found by multiplying its power spectral density by ω^2 or by ω^4 , respectively. This rule is quite general [8] and we shall use it without further comment in what follows to obtain the power spectral densities of $\ddot{\theta}(t)$, and $\ddot{Y}(t) + z\ddot{\theta}(t)$

The constants which define the vehicle and track enter the spectral densities of interest in a fairly complex manner. Further, the measures of ride roughness that are being employed involve these power spectral densities either through an integral (variance) or a deriva-

tive (peak value). Hence, we shall not employ analytical methods to perform quantitative parameter studies on ride roughness but instead use numerical methods.

A definite formula for $P_{Y_0}(\lambda)$ is now required

$$(5.7) \qquad P_{Y_0}(\lambda) = \frac{\sigma_{Y_0}^2}{\lambda_0\sqrt{2\pi}} e^{-\frac{\lambda^2}{2\lambda_0^2}}$$

The quantity $\sigma_{Y_0}^2$ is the variance of $Y_0(x)$; λ_0 is a constant which may be used to adjust the spread of the single peak possessed by $P_{Y_0}(\lambda)$. This formula appears to give a reasonable fit to a power spectral density which we obtained from ground measurement. Further measurements may well suggest different expressions for the right hand side of (5.7)*. However, for the present, it is a reasonable selection.

A set of dimensionless parameters we have found convenient to employ in numerical calculations is the following:

$$(5.8) \qquad \begin{cases} \alpha = \dfrac{l\omega_y}{v_0} , \quad \beta = \dfrac{\lambda_0 v_0}{\omega_y} , \quad \nu = \dfrac{v_0}{v} , \\ \dfrac{a}{l} , \quad \zeta_y , \quad \bar{\kappa} = \dfrac{l}{2\rho} , \end{cases}$$

where v_0 is a reference speed and ρ is the radius of gyration about the c. g. of the frame. In terms of these parameters, the first of (4.17) can be rewritten as

* The covariance corresponding to (5.7) is analytic; thus, the r. f. having this covariance is m. s. analytic [8 Ch. 7] While m. s. analyticity has certain predictibility implications, we shall ignore them.

$$(5.9)\quad \frac{P_y(z,\omega)_1}{\sigma_{Y_0}^2/\omega_y} = \frac{1}{2}\left[(1-r^2)^2+(2\zeta_y r)^2\right]^{-1}\left[\left(1-\frac{r^2}{K^2}\right)^2\right]^{-1}\left\{(1\right.$$

$$+\cos\alpha\nu r)\left[1+(2\zeta_y r)^2\right]\left[\left(1-\frac{r^2}{K^2}\right)^2+(2\zeta_y r)^2\right]$$

$$+8\zeta_y r^3\frac{z}{l}\sin\alpha\nu r\left[\left(\frac{r^2}{K^2}-1-4\zeta_y^2 r^2\right)-\frac{1}{K^2}(r^2-1-4\zeta_y^2 r^2)\right]$$

$$\left.+4\frac{z^2}{l^2}(1-\cos\alpha\nu r)\left[1+(2\zeta_y r)^2\right]\left[(1-r^2)^2+(2\zeta_y r)^2\right]\right\}\frac{\nu}{\beta\sqrt{2\pi}}e^{-\frac{r^2\nu^2}{2\beta^2}},$$

for example. The other spectral densities can be put in a similar form.

As mentioned in the Introduction, the two ride criteria which we shall employ are either the magnitude of the largest peak in the power spectral density or the value of the variance for acceleration. The accelerations of the frame we shall consider are $\ddot{Y}(t)$, $\ddot{Y}(t)+\frac{l}{2}\ddot{\theta}(t)$ and $\ddot{\theta}(t)$; the first is vertical acceleration at the c. g., the second is vertical acceleration over idealized wheel-tire, and the third is pitching acceleration.

Only the two special cases of the vehicle shown in Figure 1 will be considered at this time. The formulas needed for the power spectral densities are obtained from either (4.17) or (4.18). Ordinate values for vertical acceleration results are divided by $\sigma_{Y_0}^2\ \omega_y^4$ and those for angular acceleration results by $\sigma_{Y_0}^2\omega_\theta^4/l^2$ these factors will be omitted on the graphs. We shall also omit the subscripts 1 or 2 on the ordinate designation, since the subscripts 1 and 2 refer to the cases $a/l=0$ and $a/l\neq 0$, respectively, and these are indicated explicitly upon the graphs.

Figures 2-4 present numerical results on the maximum value of the power spectral densities and on the variances of $\ddot{Y}(t)$, $\ddot{Y}(t)+z\ddot{\theta}(t)$ and $\ddot{\theta}(t)$ as functions of the parameters α and a/l

It is immediately clear that results on variance with $a/l=0$ and $a/l = 0.0625$ and also on magnitude of the maximum value of the power spectral density do not differ appreciably until α becomes larger than 4 or 5. We note that $a/l = 0.0625$ corresponds to a total foot print length--rear plus front-- of 25% of the wheel base length l; most wheeled vehicles have a value of approximately 10%. Thus, a/l does not appear to be a significant parameter with the ride criteria we are employing unless α exceeds 4 or 5. In what follows, therefore, it is convenient to set $a/l=0$.

On the other hand, changes in the parameter α have a pronounced influence on the magnitude of the largest peak in the power spectral density and the variance of the various accelerations. This strong influence is due to the fact that the location of the bands on the ω-axis which the trigonometric terms containing α filter change with α, sometimes reducing peaks sharply and sometimes reinforcing them.

For the vertical acceleration, there is a unique value of α which makes the peak magnitude and variance a minimum. Thus, if the best ride under the conditions assumed is determined either by the minimum peak magnitude in the spectral density or by the minimum value of variance, we see that for each point on the frame there is a unique value α which provides such a best ride; the range of α values being from .4088 to 2.6150. We also note, however, that slight changes from these α values produced large changes in minimum peak magnitude and variance, i. e. the best ride is a sharply tuned function of α. For too large a value of α with $z=\frac{l}{2}$ the variance is larger than with a very small α; but with $z=0$, the variance is always reduced below its value at $\alpha=0$ by increasing α.

The peak value in the spectral density and the variance for the pitching acceleration exhibit a difference dependence upon α than encountered with the vertical acceleration, as may be seen from Figure 4. Each is zero when α equals zero; as α increases each reaches a maximum value, falls to a minimum and then oscillates. The value of α which gives the best ride in vertical acceleration does not give a best ride in pitch. However, it must be remembered that the parameter $\bar{r}$ is adjustable. Hence, it may be possible to select α and $\bar{r}$ to give a best ride from the point of view of vertical acceleration and pitch, or, at least, a good compromise to a best ride in both cases; this point is now under investigation.

The ride criterion used in Figure 5 is the magnitude of the largest peak in the power spectral density of $\ddot{Y}(t)$. With $\frac{a}{l}=0$, $\beta=1.00$ $\nu=1.00$, the value of α which makes the peak a minimum is plotted as a function of ζ_y (See Figure 5a). In addition, the minimum peak value is plotted. Although the range of the minimum peak is not large for the ζ_y considered, there is a unique ζ_y ($=.6953$) and α ($=1.9170$) which makes this minimum peak a minimum. Thus, one pair of ζ_y and α give the best ride in the peak of power spectral density sense. Figure 5b is a graph of the peak in the power spectral density of $\ddot{Y}(t)$ as a function of ν with $\frac{a}{l}=0$, $\beta=1.00$, $\zeta_y=.6953$, $\alpha=1.9170$. A $\nu>1$ corresponds to $v<v_0$ and $\nu<1$ corresponds to $v>v_0$. Thus, as speeds decrease below v_0, which may be looked upon as design speed, the peak value falls gradually; however, as v increases above v_0 the peak value rises sharply, i. e. the ride rapidly gets worse.

The results presented in Figure 6 correspond to results given in Figure 5 except that the ride criterion is now the value of the variance of $\ddot{Y}(t)$. Figures 7 and 8 correspond to Figures 5 and 6 but for $\ddot{Y}(t)+\frac{l}{2}\ddot{\theta}(t)$. The conclusions drawn from Figure 5 are reflected in these three Figures.

Figures 8 and 9 present results on $\ddot{\theta}(t)$ that is the same as given in Figures 5 and 6, respectively, for $\ddot{Y}(t)$. However, with $\ddot{\theta}(t)$ neither the minimum variance nor minimum peak value has a minimum value as a function of ζ_y. This result agrees with the observation that the oscillations of $\ddot{\theta}(t)$ may be reduced to zero by making l infinite and using a rigid suspension system. For arbitrarily selected ζ_y, these two Figures again show the great sensitivity of variance and maximum peak value of power spectral density to changes in ν in the neighborhood of $\nu = 1$.

The parameter β was not changed in the studies reported in this paper. We recall that β contains λ_0 which is a measure of the spread of $P_{Y_0}(\lambda)$. At this time, we have no numerical value for λ_0 moreover, we are not certain that (5.7) is the most reasonable choice for $P_{Y_0}(\lambda)$. Hence, it does not seem appropriate to obtain results with other than one value of β. It is worthy of note, however, that since

$$\alpha\beta = \lambda_0 l , \tag{5.10}$$

if we had a numerical value for λ_0 and the α associated with a best ride in same sense for a given ζ_y, β, etc., then a numerical value for the corresponding wheel base length could be obtained.

The results presented in this paper represent the beginnings of an attack on the interesting problem of how to design a vehicle to achieve higher speeds than obtained at present under off-road conditions. A large effort will surely have to be devoted to obtaining a statistical characterization of ground contour geometry and quantitative ride criteria before either the method of attack proposed can be justified here and definitive design results can be obtained.

REFERENCES

1. J. L. Bogdanoff and F. Kozin. Behavior of a Linear One-Degree of Freedom Vehicle Moving with Constant Velocity on a Stationary Gaussian Random Track, Report No. 48, Land Locomotion Laboratory, Detroit, February 1959.

2. F. Kozin and J. L. Bogdanoff. On the Statistical Analysis of the Motion of Some Simple Two-Dimensional Linear Vehicles Moving on a Random Track, International Journal of Mechanical Sciences, Vol. 2, No. 3, p. 168, 1960.

3. D. E. Goldman, Effects of Vibration on Man, Handbook of Noise Control, McGraw-Hill, New York, 1957.

4. H. E. von Gierke, Transmission of Vibratory Energy Through Human Body Tissue, Proc. of First Nat. Biophysics Conf., 1957, Yale University Press, 1959.

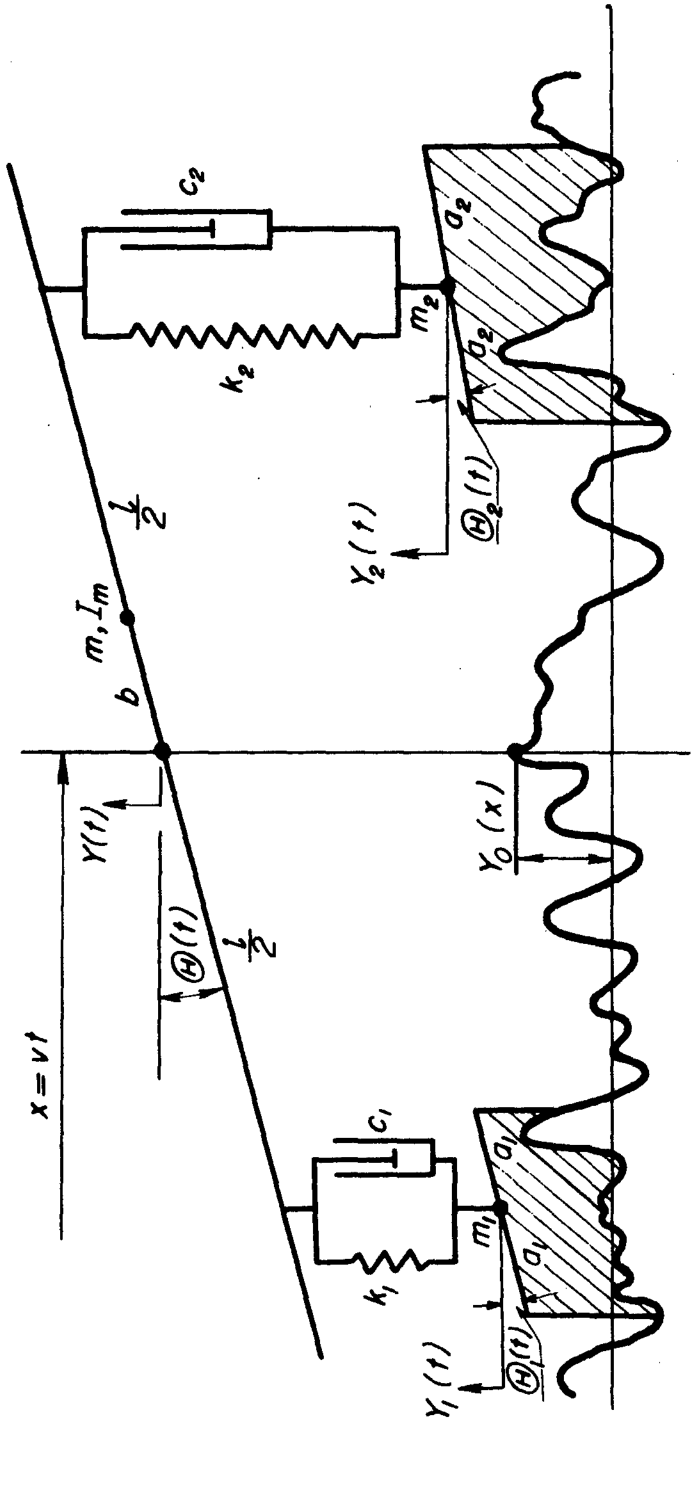
5. R. R. Coermann, The Mechanical Impedance of the Human Body in the Sitting and Standing Position, etc., 3rd Annual Meeting of the Biophysical Society, Pittsburgh, Penn., Feb. 1959.

6. F. Kozin and J. L. Bogdanoff, On the Statistical Properties of the Ground Contour and its Relation to the Study of Land Locomotion, See preceeding paper.

7. J. E. Moyal, Stochastic Processes and Statistical Physics, Journal of Royal Statistical Society, Sec. B., Vol. 11, p. 150, 1950.

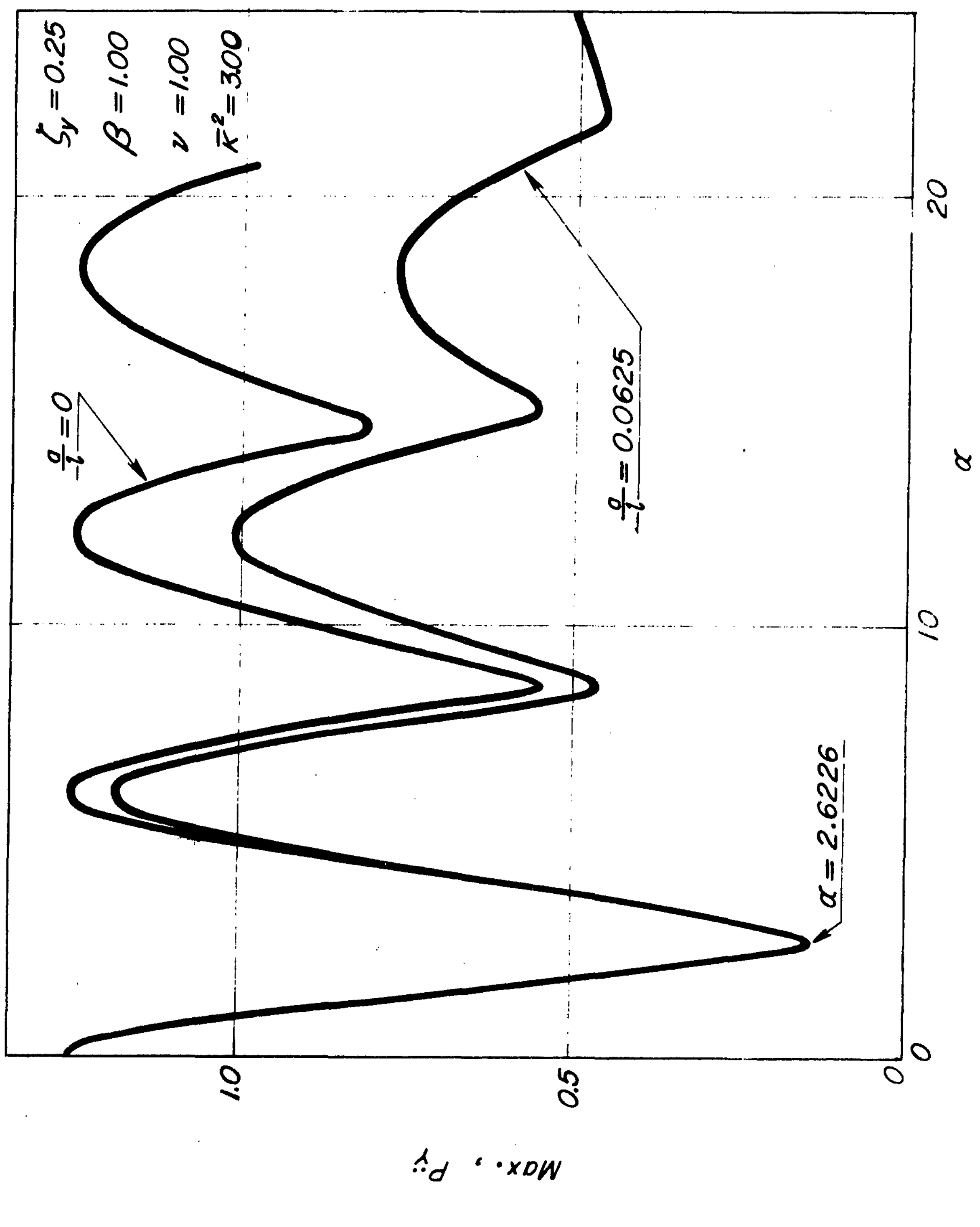
8. M. S. Bartlett, An Introduction to Stochastic Processes, Cambridge University Press, Cambridge, 1955.

9. S. O. Rice, Mathematical Analysis of Random Noise, Bell System Tech. Journal, Vol. 23, p. 282, 1944 and Vol. 24, p. 46, 1945.



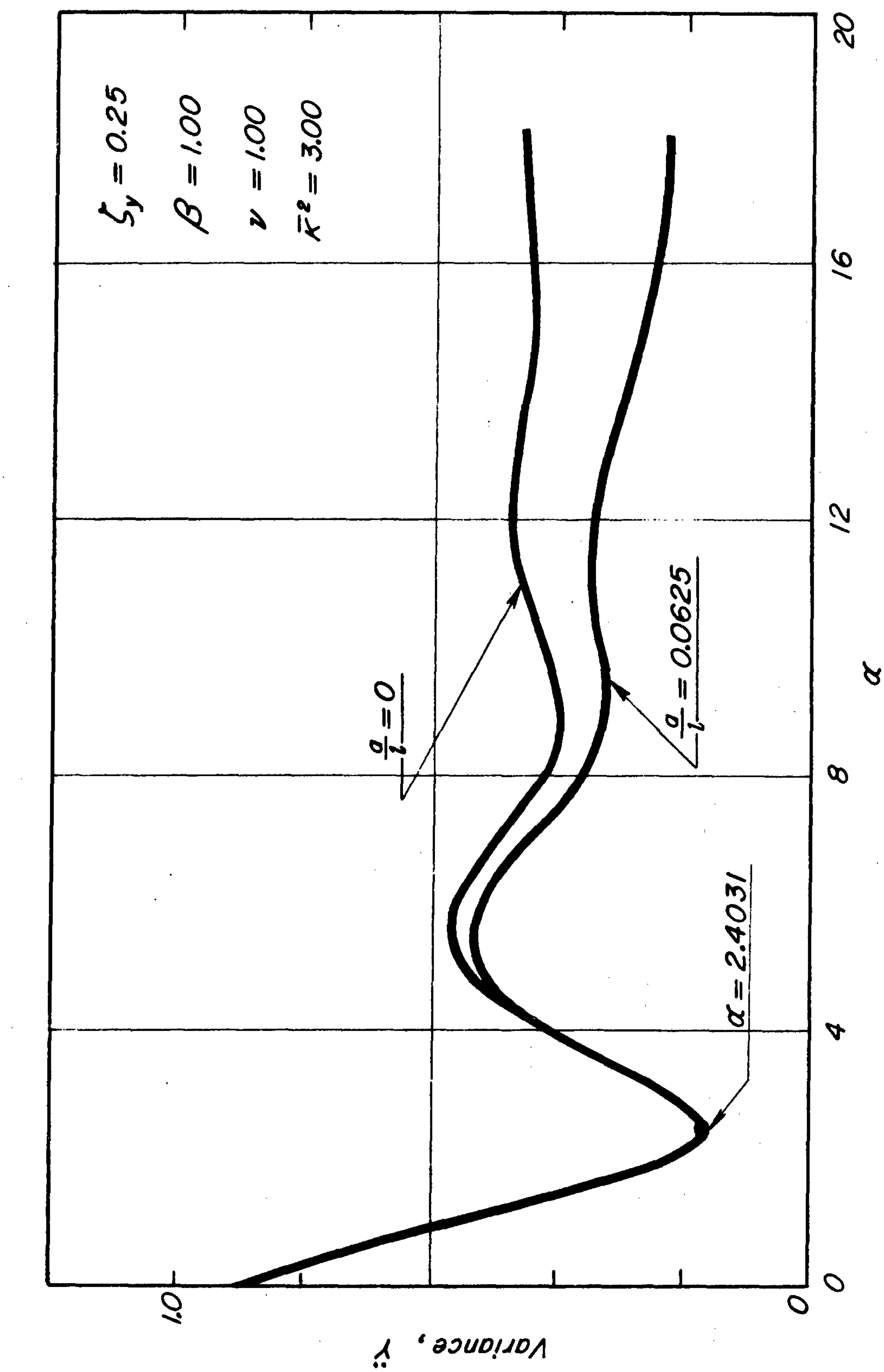
x = vt
Y(t)
Θ(t)
l/2
b
m, I_m
l/2
c_2
k_2
m_2
a_2
a_2
Θ_2(t)
Y_2(t)
Y_0(x)
c_1
k_1
m_1
a_1
a_1
Θ_1(t)
Y_1(t)

Fig. 1



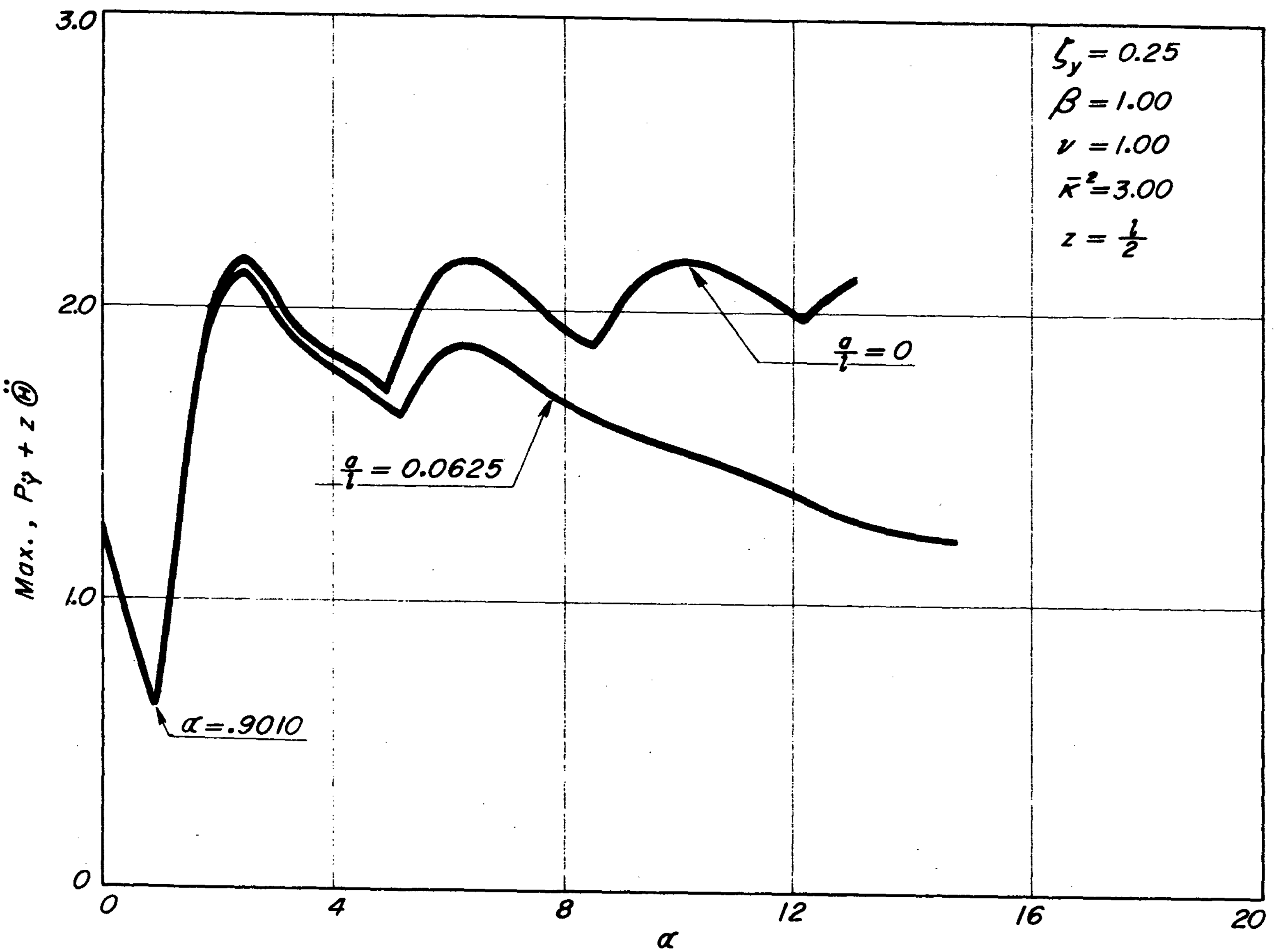
$\zeta_y = 0.25$
$\beta = 1.00$
$\nu = 1.00$
$\bar{\kappa}^2 = 3.00$
$\frac{a}{l} = 0$
$\frac{a}{l} = 0.0625$
$\alpha = 2.6226$
α
Max., P_y
0
10
20
0
0.5
1.0

Fig. 2a



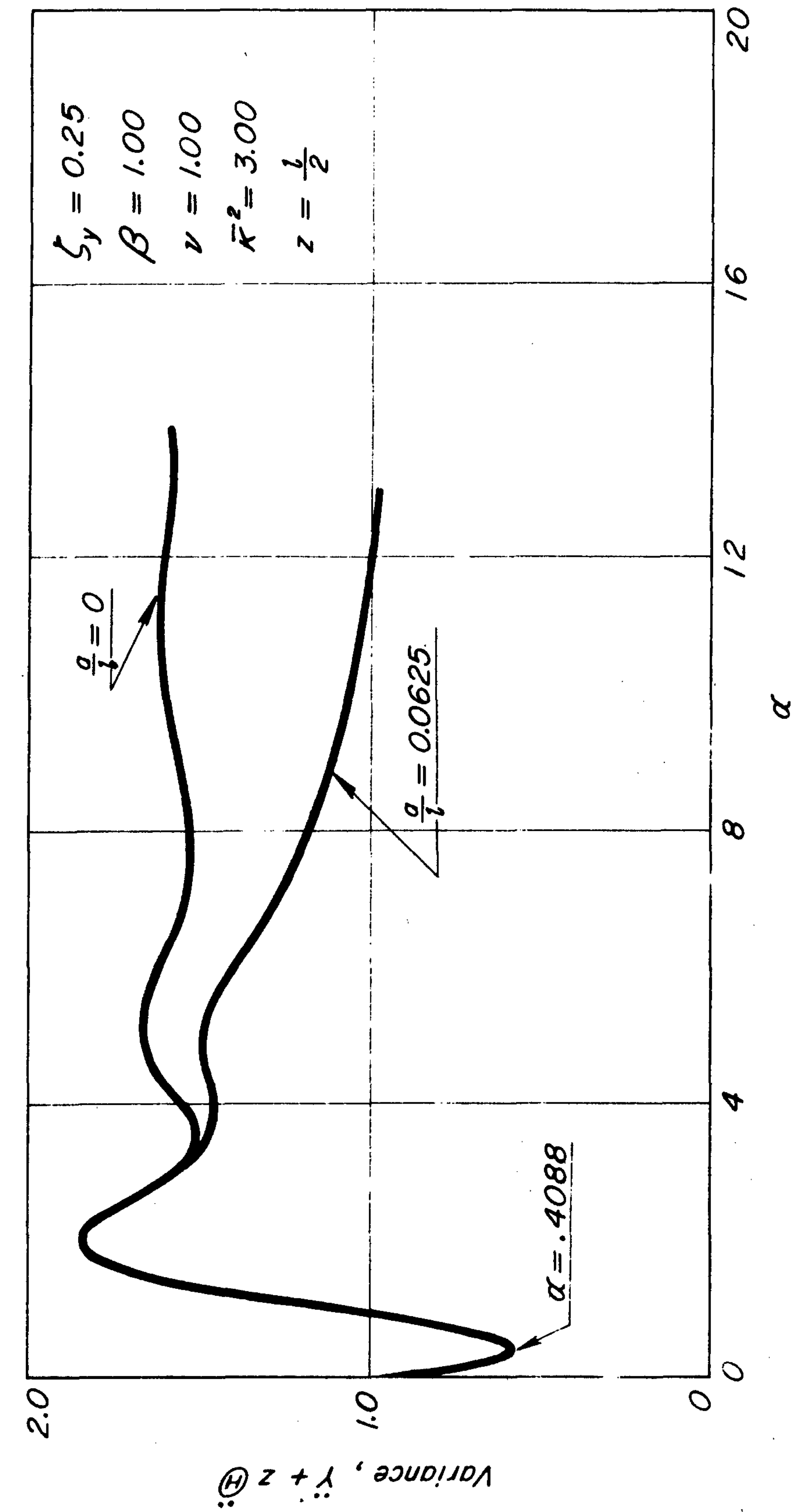
$\zeta_y = 0.25$
$\beta = 1.00$
$\nu = 1.00$
$\bar{\kappa}^2 = 3.00$
$\frac{a}{l} = 0$
$\frac{a}{l} = 0.0625$
$\alpha = 2.4031$
α
0
4
8
12
16
20
Variance, $\dot{Y}$
0
1.0

Fig. 2b



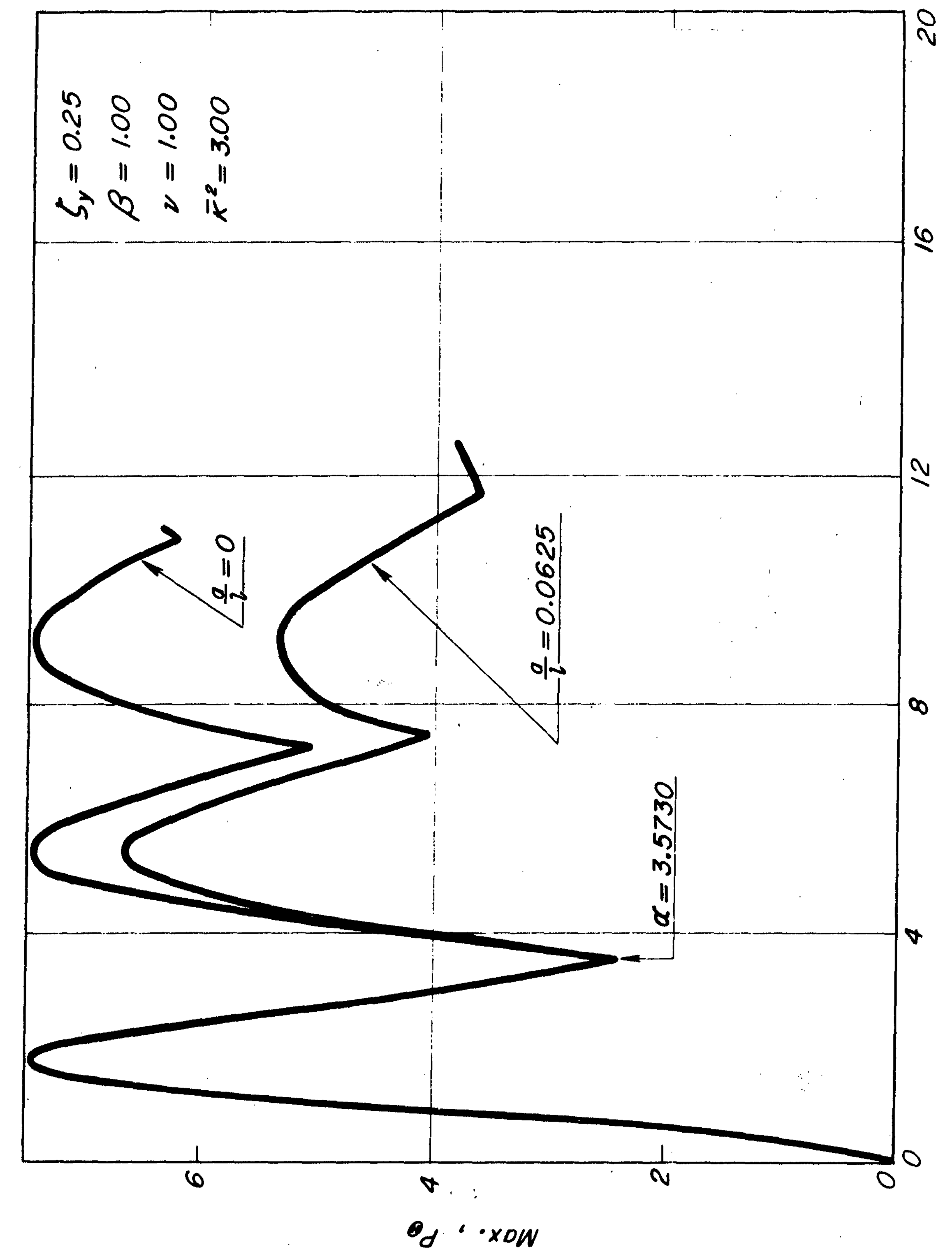
3.0
2.0
1.0
0
Max., $P_y + z\ddot{\Theta}$
0
4
8
12
16
20
α
$\zeta_y = 0.25$
$\beta = 1.00$
$\nu = 1.00$
$\bar{\kappa}^2 = 3.00$
$z = \frac{l}{2}$
$\frac{a}{l} = 0$
$\frac{a}{l} = 0.0625$
$\alpha = .9010$

Fig. 3a



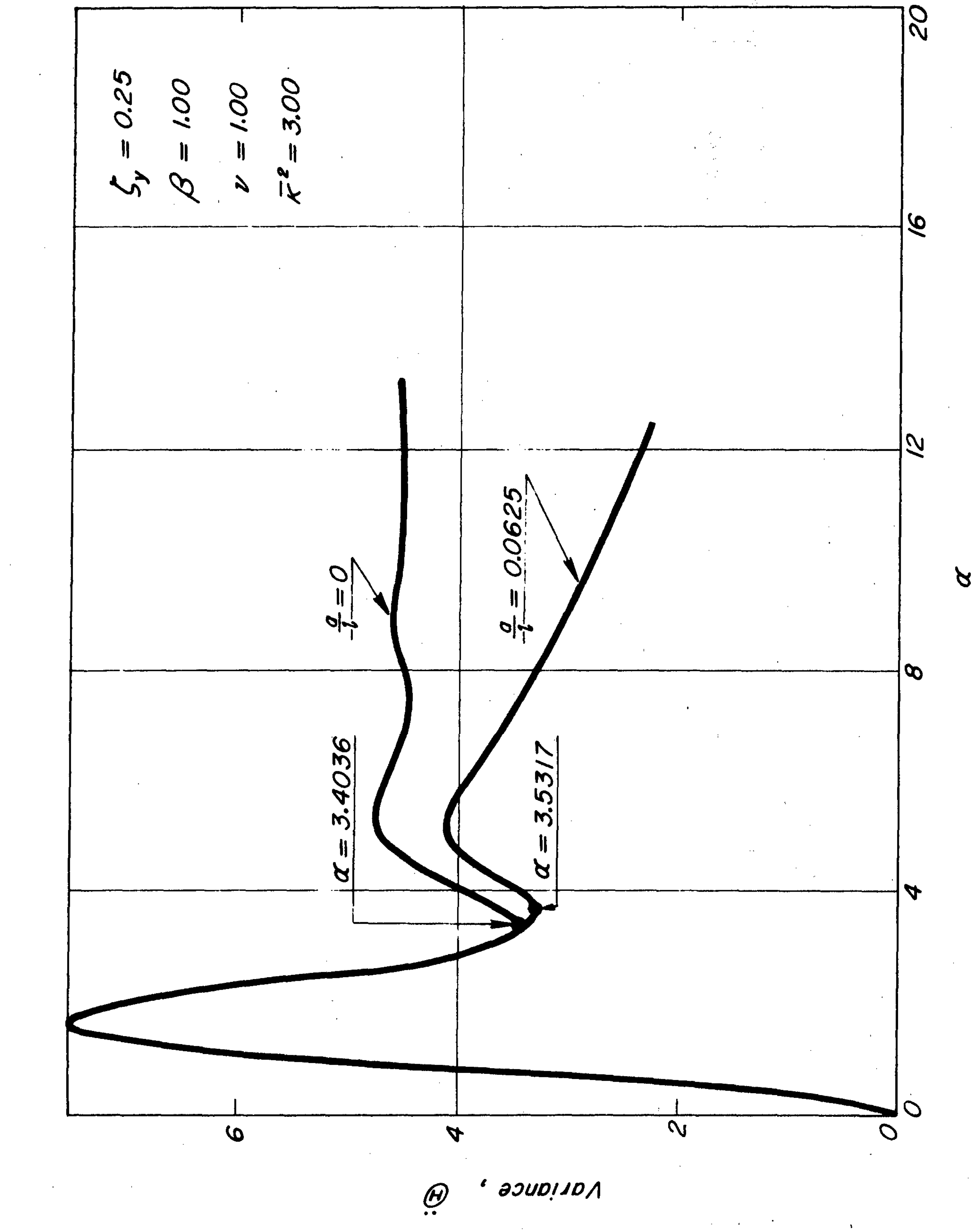
$\zeta_y = 0.25$
$\beta = 1.00$
$\nu = 1.00$
$\bar{\kappa}^2 = 3.00$
$z = \frac{1}{2}$
$\frac{a}{l} = 0$
$\frac{a}{l} = 0.0625$
$\alpha = .4088$
0
4
8
12
16
20
α
0
1.0
2.0
Variance , $\ddot{Y} + z\ddot{\Theta}$

Fig. 3b



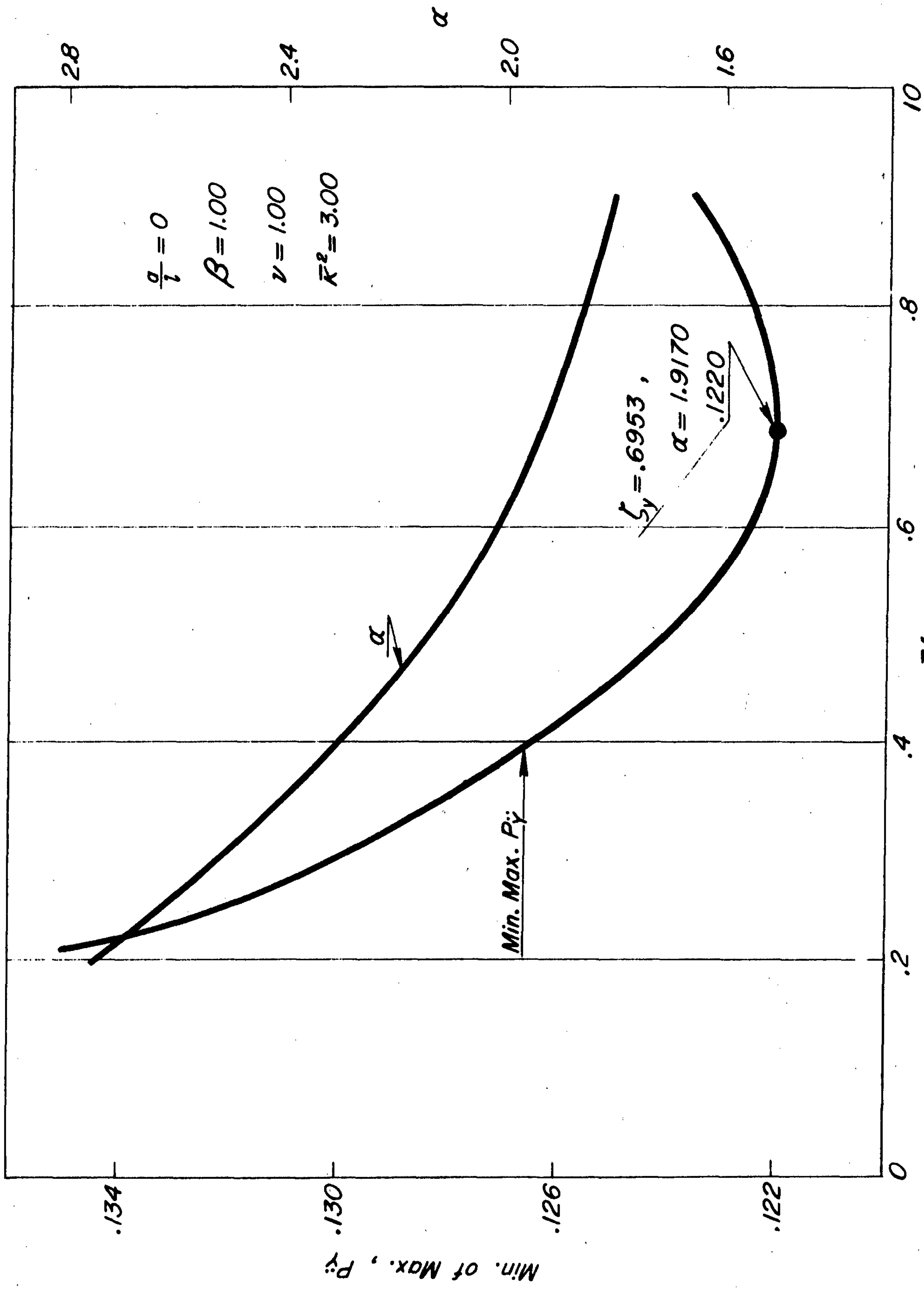
ζ_y = 0.25
β = 1.00
ν = 1.00
κ̄² = 3.00
a/l = 0
a/l = 0.0625
α = 3.5730
0
4
8
12
16
20
α
0
2
4
6
Max., p_θ

Fig. 4a



$\zeta_y = 0.25$
$\beta = 1.00$
$\nu = 1.00$
$\bar{\kappa}^2 = 3.00$
$\frac{a}{l} = 0$
$\frac{a}{l} = 0.0625$
$\alpha = 3.4036$
$\alpha = 3.5317$
0
4
8
12
16
20
α
0
2
4
6
Variance, Θ

Fig. 4b



$\frac{a}{l} = 0$
$\beta = 1.00$
$\nu = 1.00$
$\bar{K}^2 = 3.00$
$\zeta_y = .6953$,
$\alpha = 1.9170$
.1220
α
Min. Max. P_y
2.8
2.4
2.0
1.6
.134
.130
.126
.122
0
.2
.4
.6
.8
1.0
ν
Min. of Max., P_y

Fig. 5a

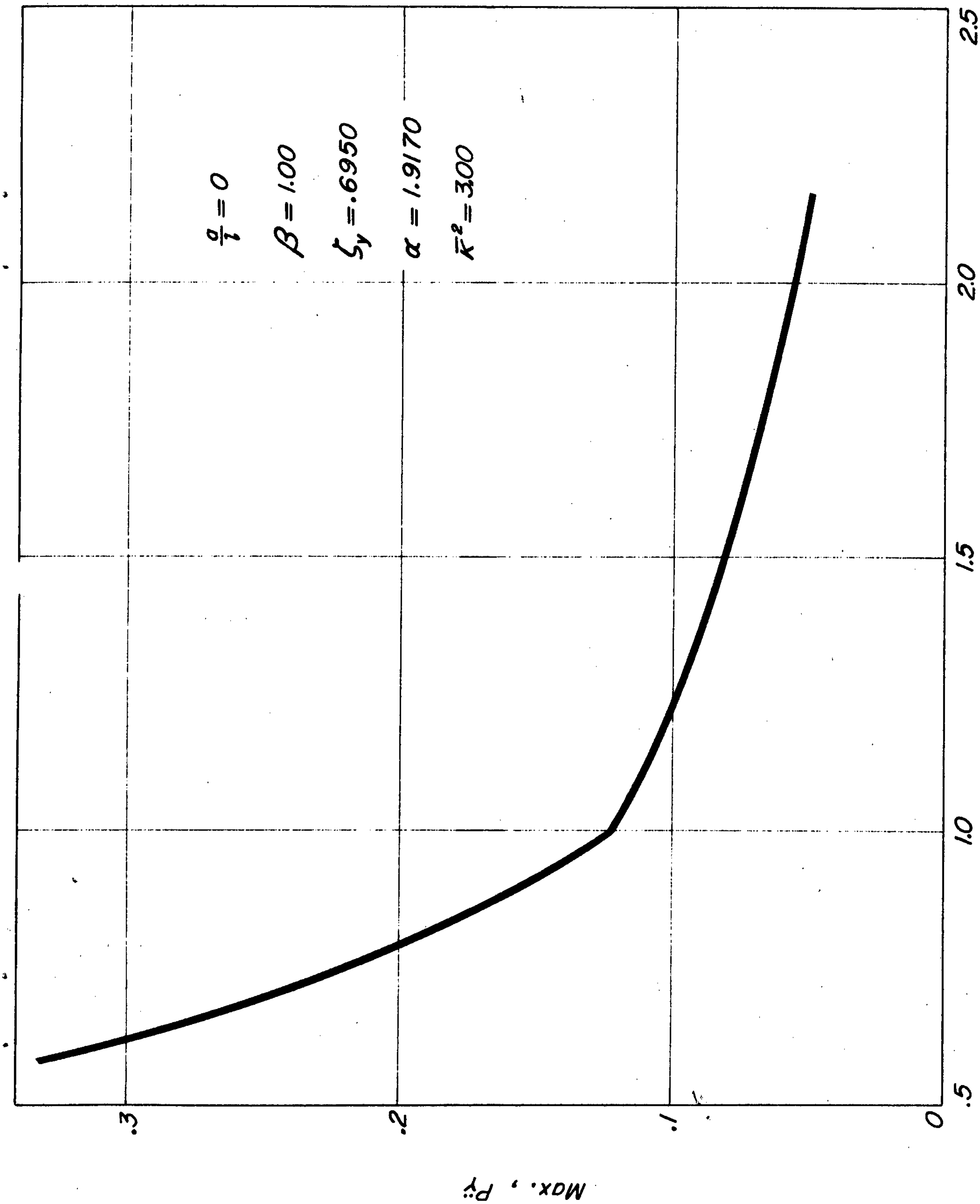
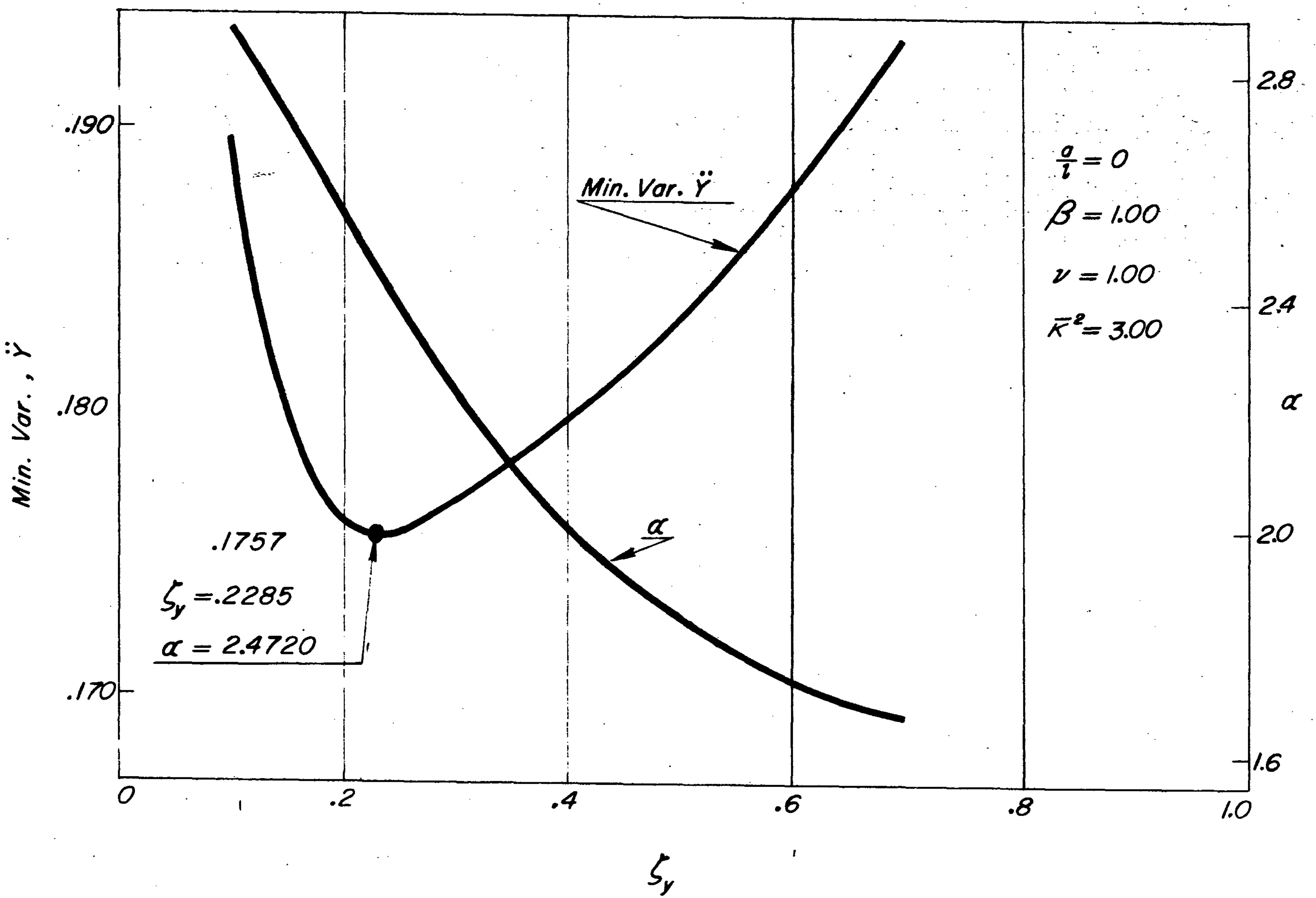


Fig. 5b



Min. Var. $\ddot{Y}$
$\frac{a}{l} = 0$
$\beta = 1.00$
$\nu = 1.00$
$\bar{\kappa}^2 = 3.00$
α
.1757
$\zeta_y = .2285$
$\alpha = 2.4720$
.190
.180
.170
Min. Var. , $\ddot{Y}$
2.8
2.4
2.0
1.6
0
.2
.4
.6
.8
1.0
ζ_y

Fig. 6a

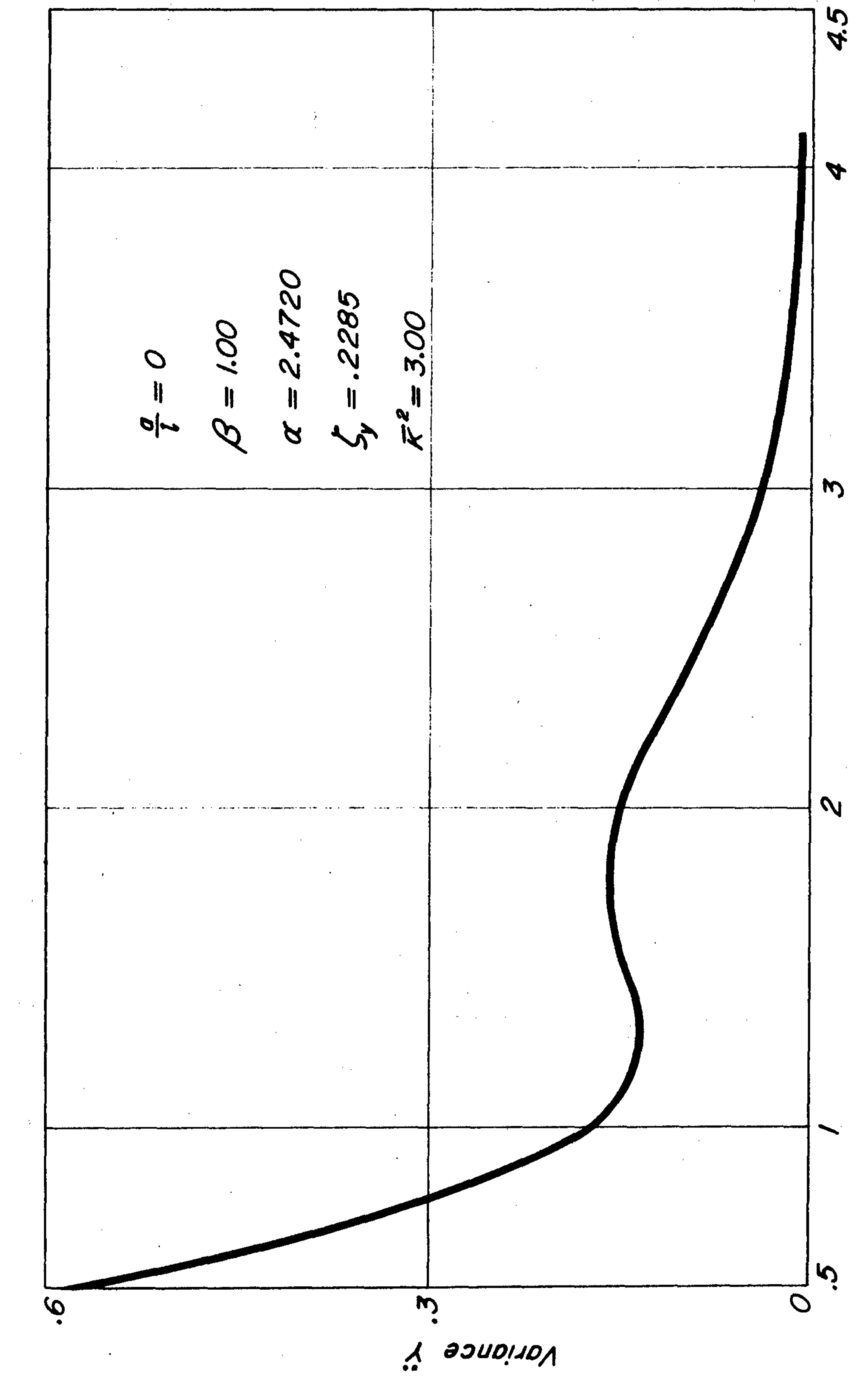
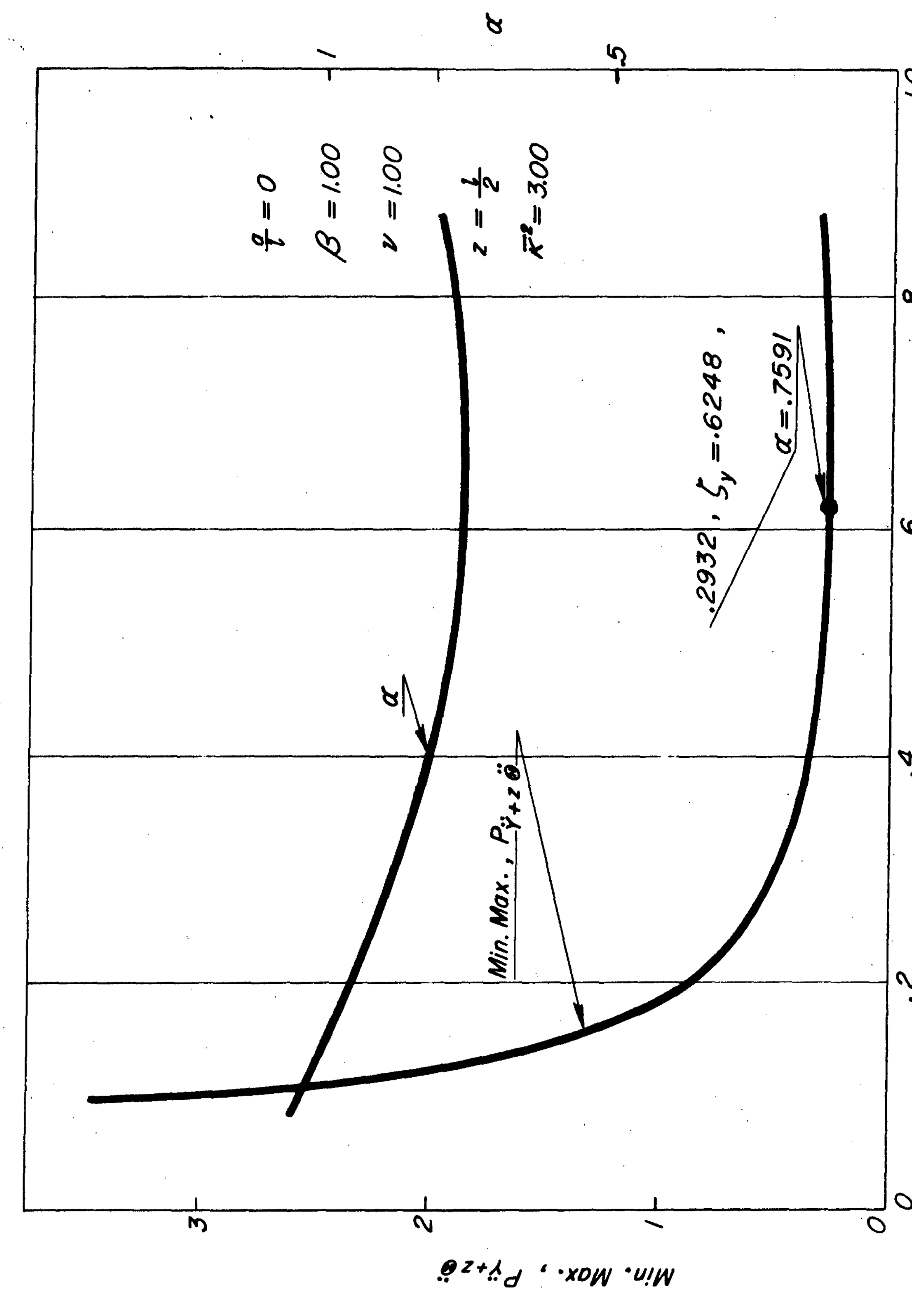

$\frac{a}{l} = 0$
$\beta = 1.00$
$\alpha = 2.4720$
$\zeta_y = .2285$
$\bar{\kappa}^2 = 3.00$
.6
.3
0
.5
1
2
3
4
4.5
ν
Variance Y

Fig. 6b



a/ℓ = 0
β = 1.00
ν = 1.00
z = 1/2
κ̄² = 3.00
.2932 , ζy = .6248 ,
α = .7591
α
Min. Max. , Pÿ+zθ̈
ζy
0
.2
.4
.6
.8
1.0
0
1
2
3
.5
1

Fig. 7a

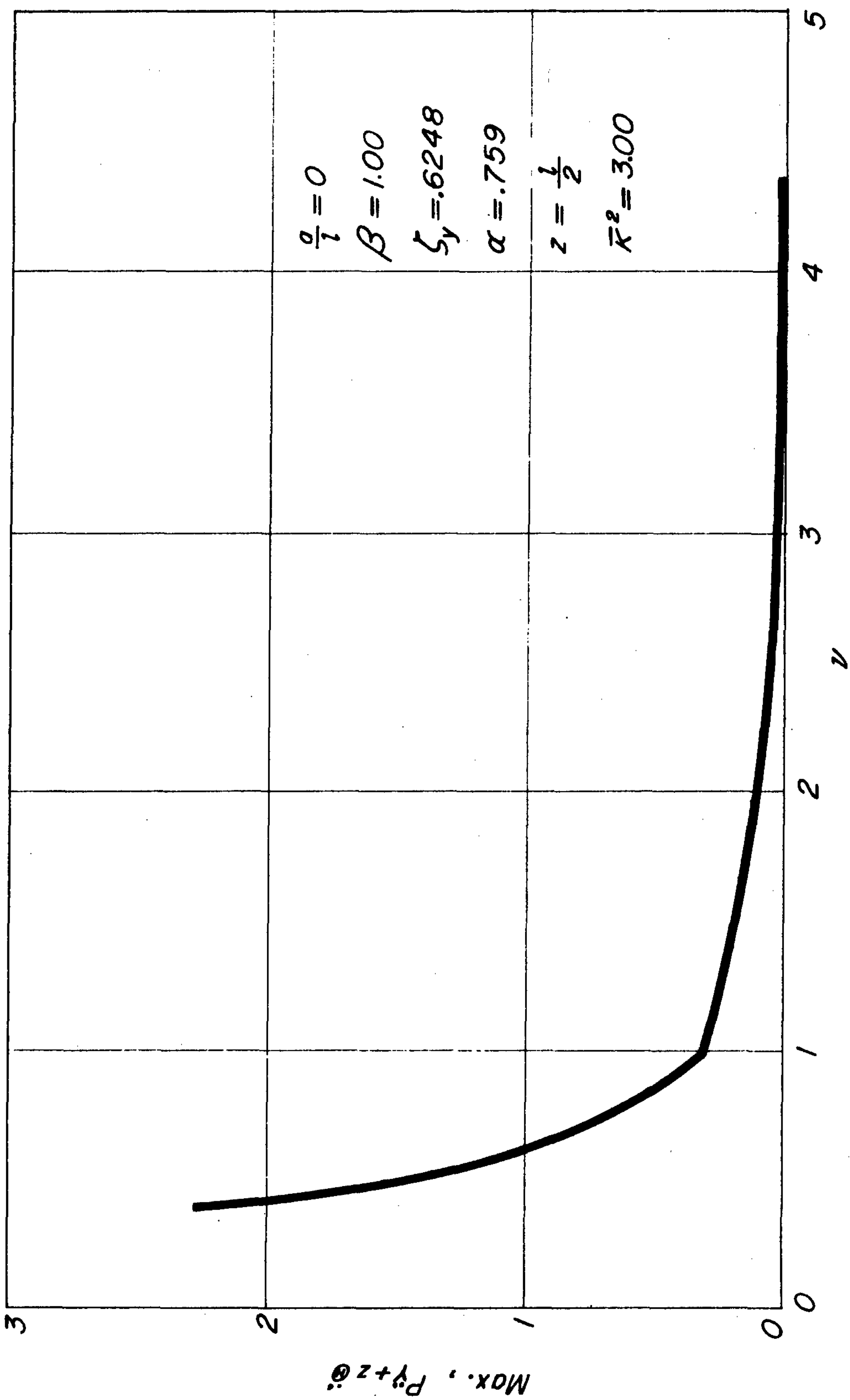
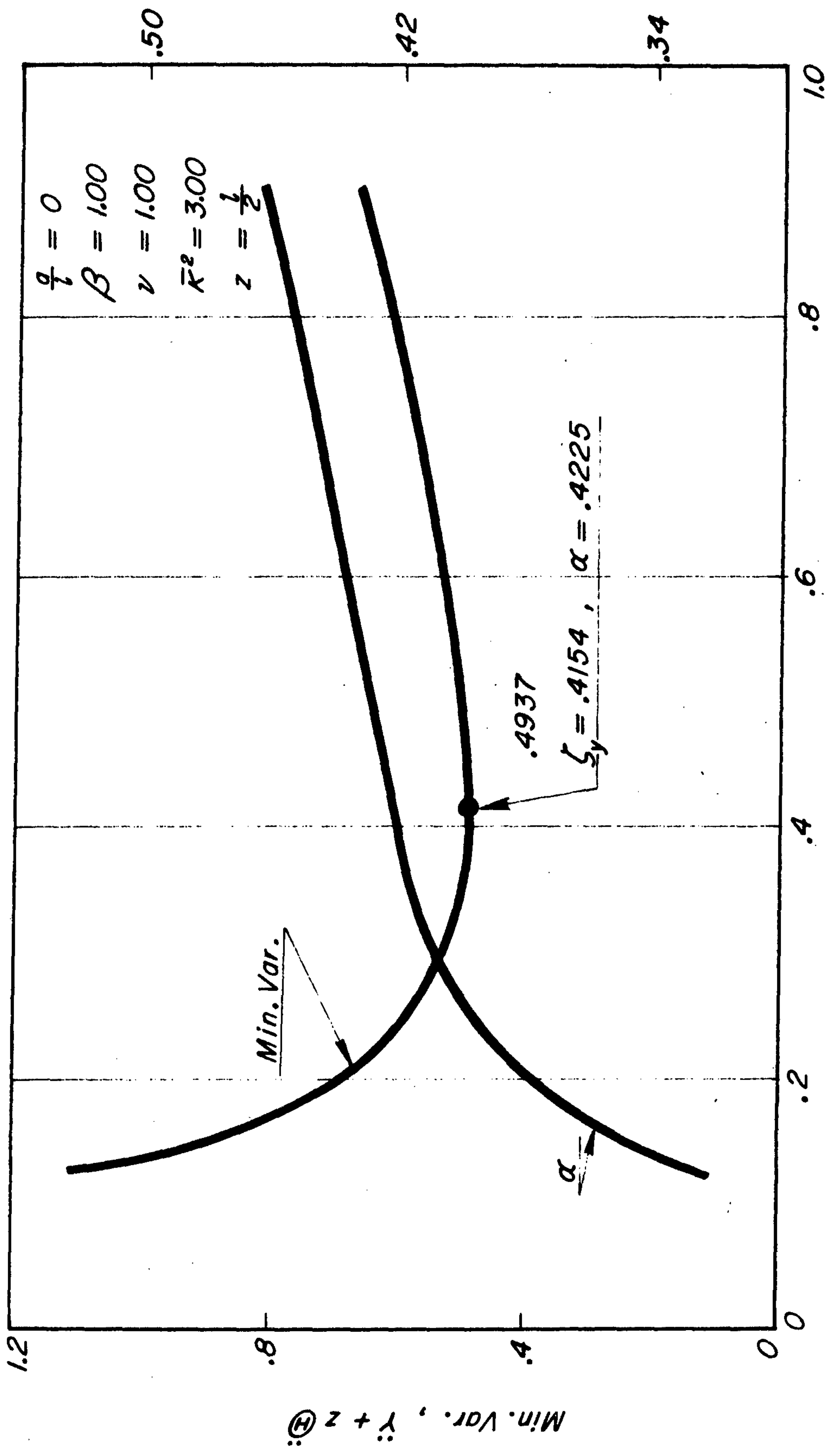


Fig. 7b



.50
.42
.34
1.2
.8
.4
0
0
.2
.4
.6
.8
1.0
$\frac{a}{l} = 0$
$\beta = 1.00$
$\nu = 1.00$
$\bar{K}^2 = 3.00$
$z = \frac{1}{2}$
Min. Var.
α
.4937
$\zeta_y = .4154$, $\alpha = .4225$
ζ_y
Min. Var. , $\ddot{Y} + z\ddot{\Theta}$

Fig. 8a

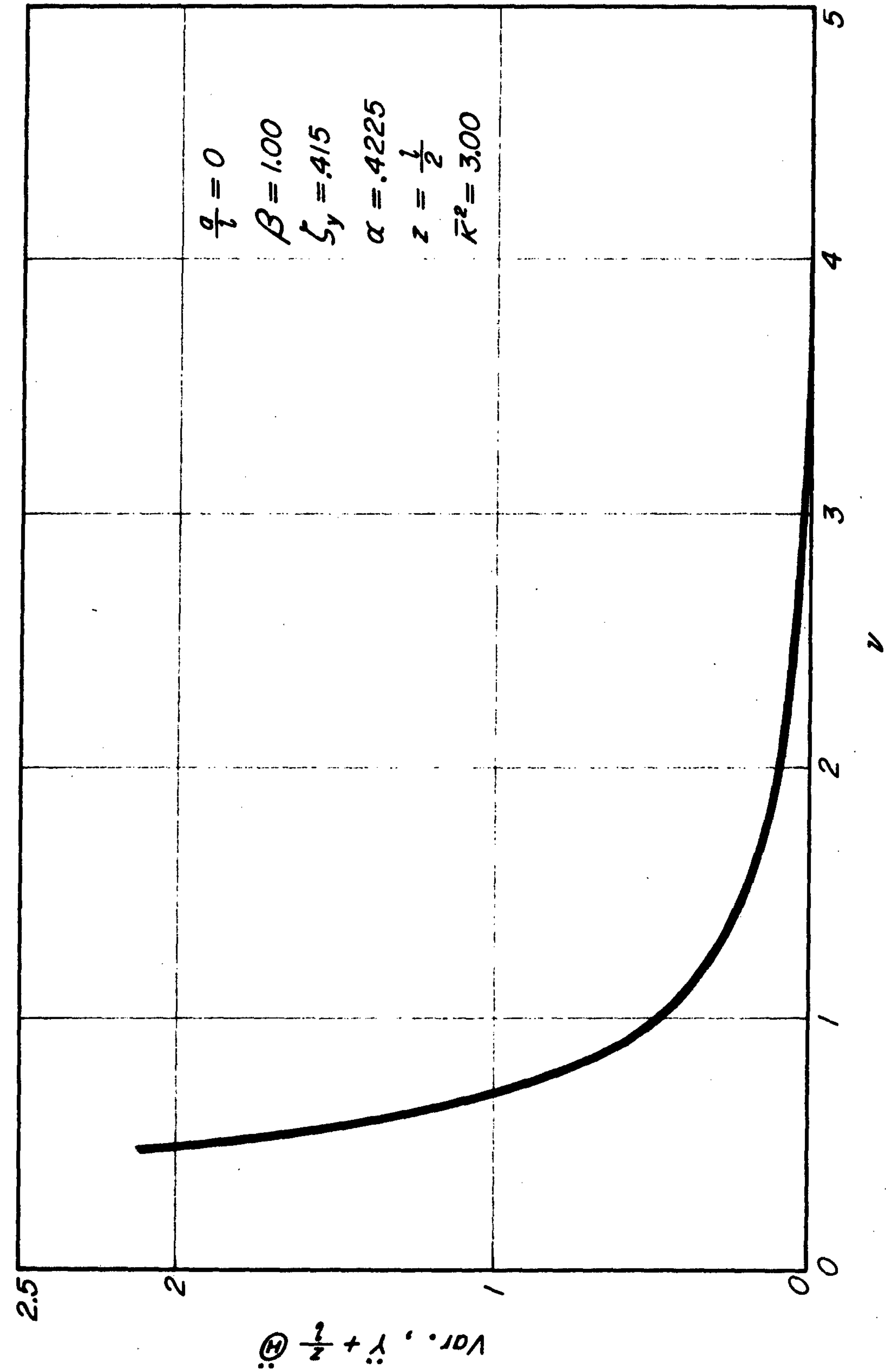
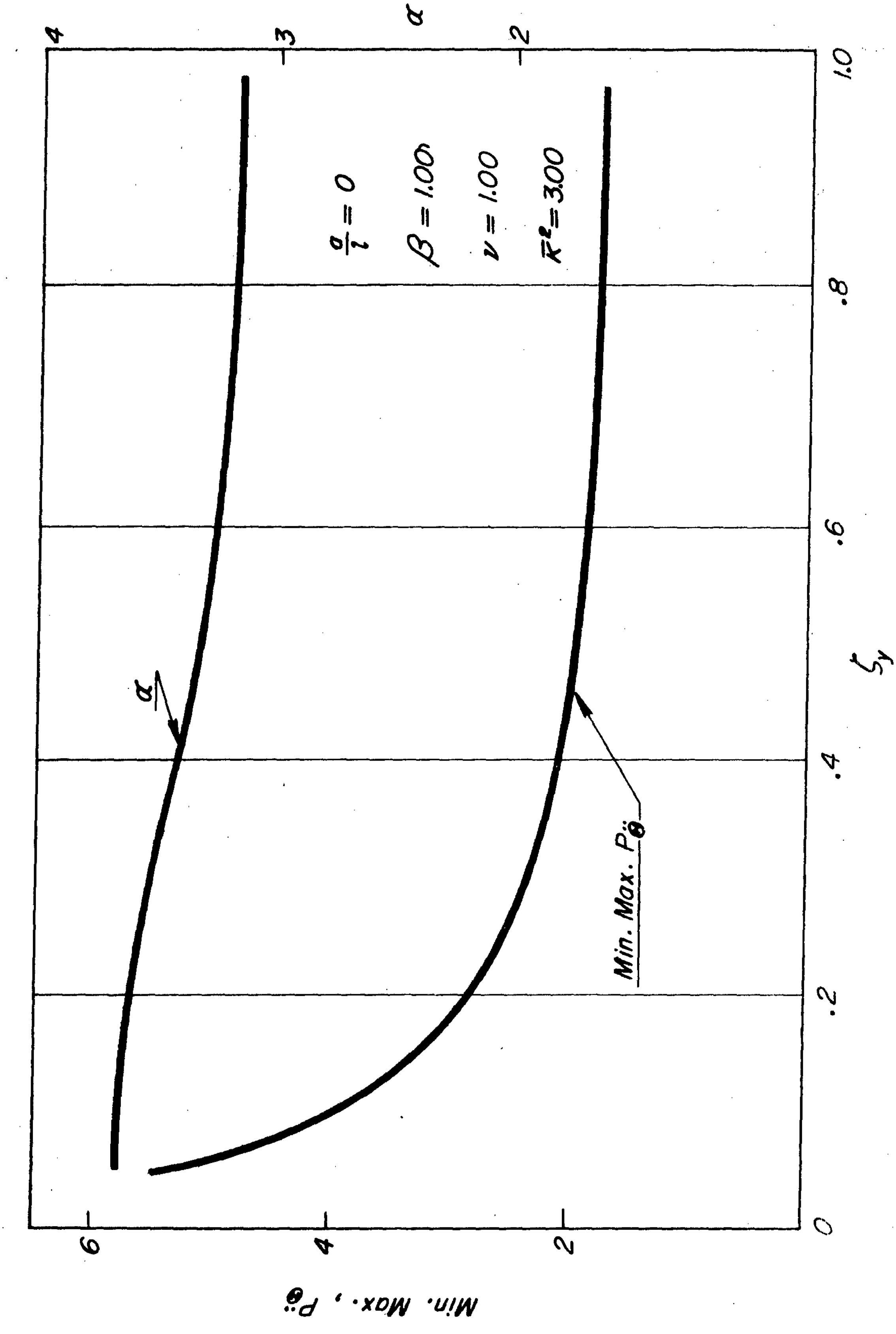


Fig. 8b



$\frac{a}{l} = 0$
$\beta = 1.00$
$\nu = 1.00$
$\bar{\kappa}^2 = 3.00$
α
Min. Max. $P_{\ddot{\theta}}$
Min. Max., $P_{\ddot{\theta}}$
ζ_y
0
.2
.4
.6
.8
1.0
2
4
6
3

Fig. 9a

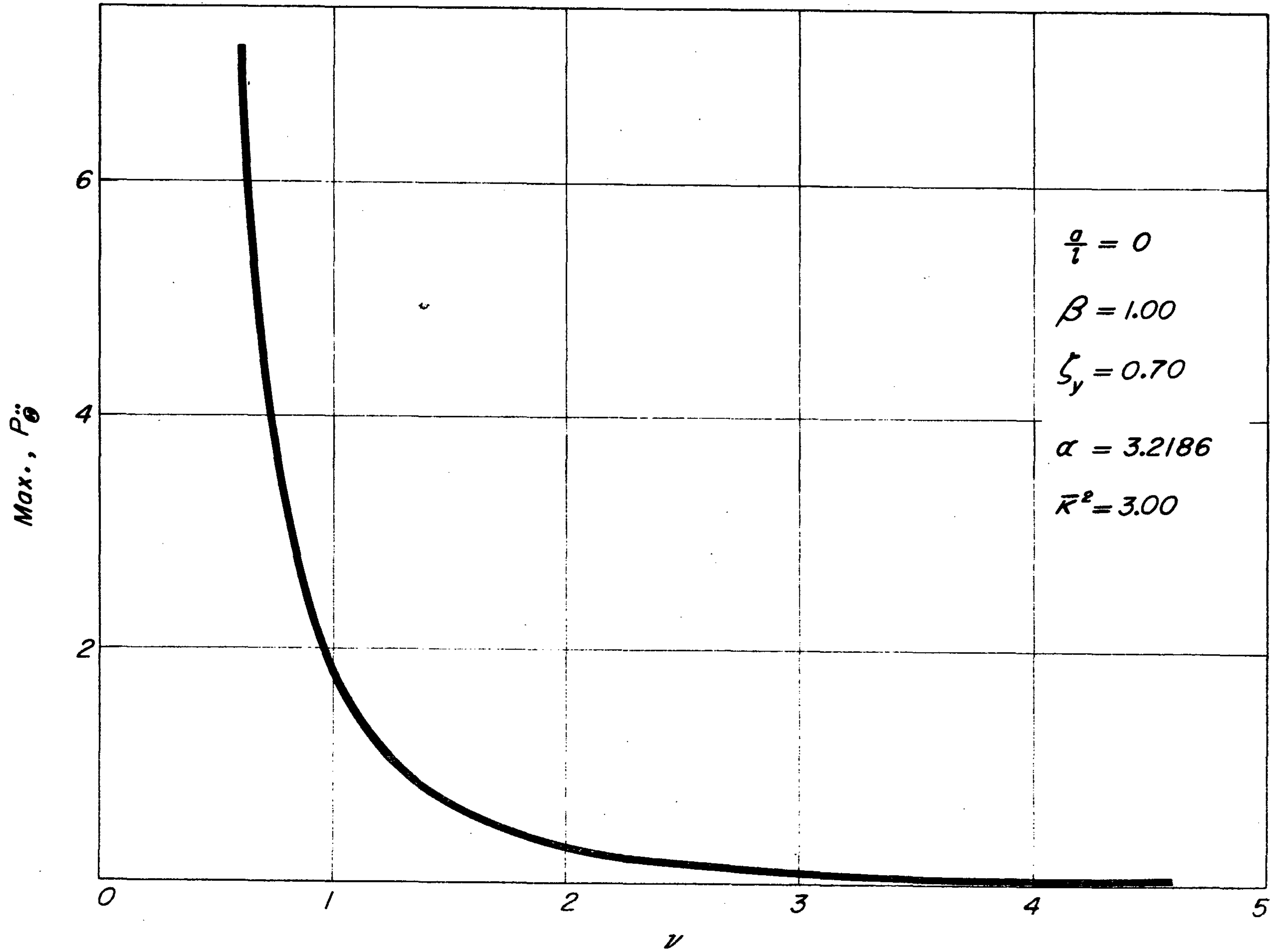
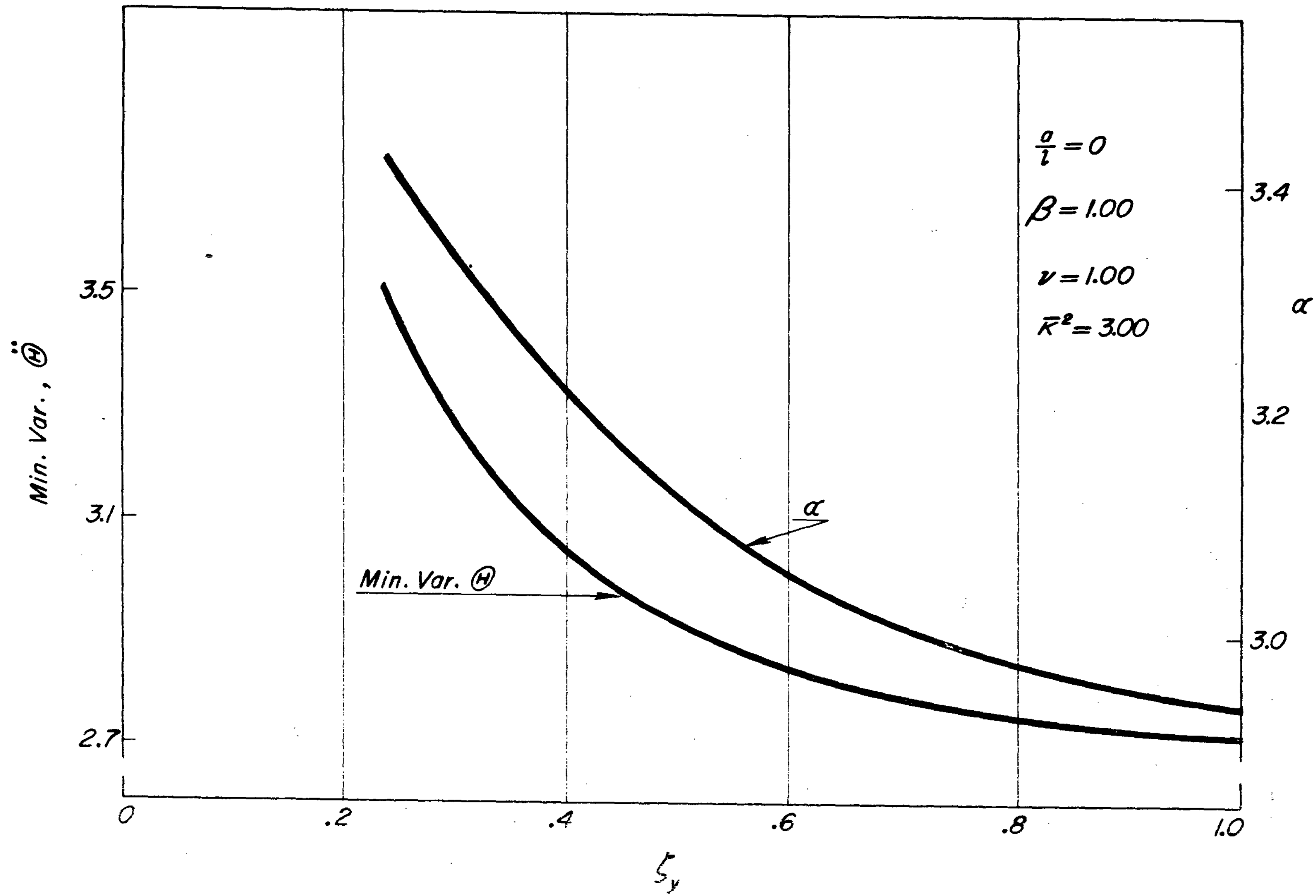
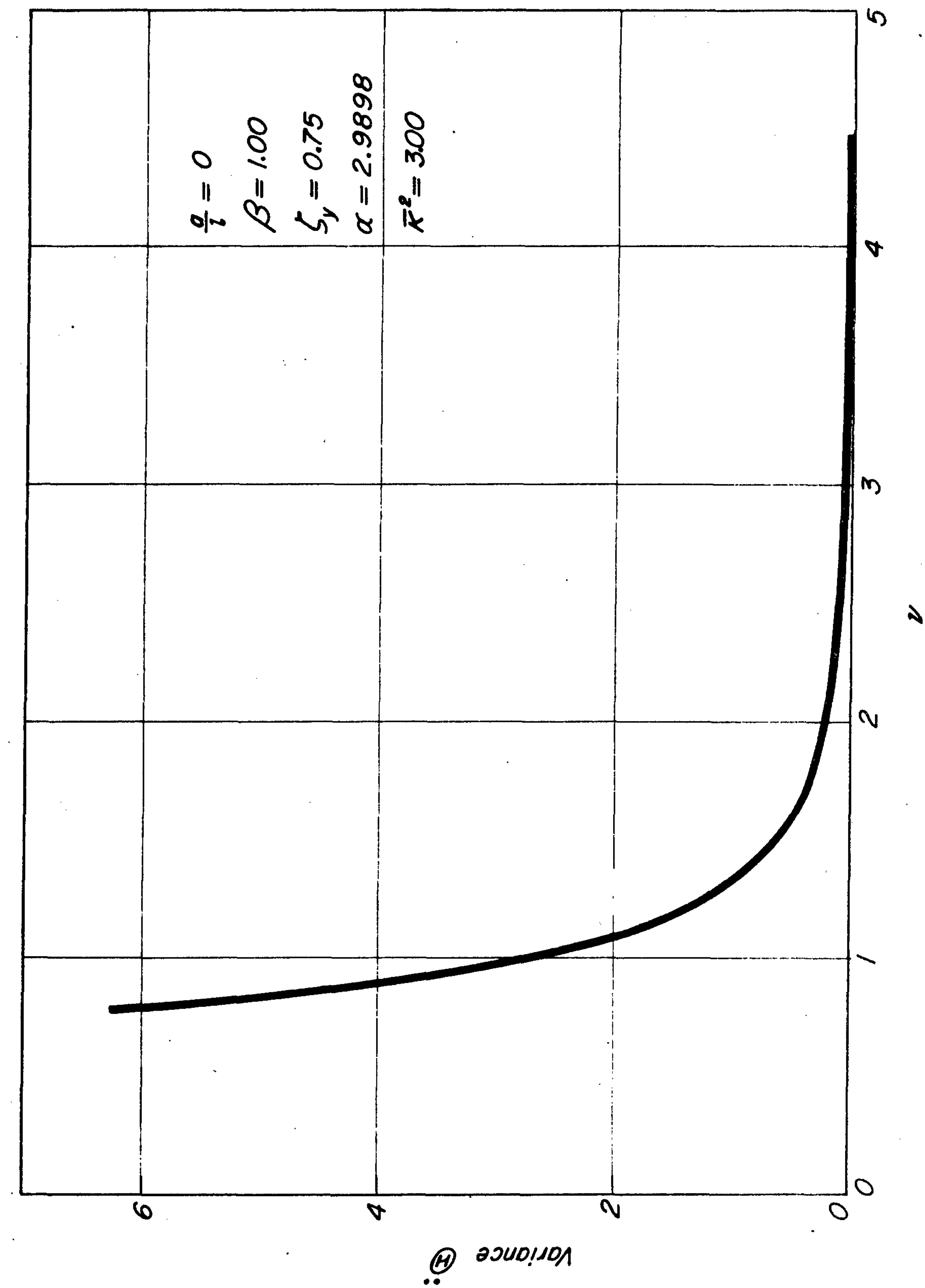

Max., $P_{\theta}^{\cdot\cdot}$
6
4
2
0
1
2
3
4
5
ν
$\frac{a}{l} = 0$
$\beta = 1.00$
$\zeta_y = 0.70$
$\alpha = 3.2186$
$\bar{\kappa}^2 = 3.00$

Fig. 9b



$\frac{a}{l} = 0$
$\beta = 1.00$
$\nu = 1.00$
$\bar{\kappa}^2 = 3.00$
α
Min. Var. Ⓗ
3.5
3.1
2.7
3.4
3.2
3.0
0
.2
.4
.6
.8
1.0
ζ_y
α
Min. Var., Ⓗ̈

Fig. 10a



$\frac{a}{l} = 0$
$\beta = 1.00$
$\zeta_y = 0.75$
$\alpha = 2.9898$
$\bar{\kappa}^2 = 3.00$
Variance $\ddot{\Theta}$
ν
0
1
2
3
4
5
0
2
4
6

Fig. 10 b

LIST OF PUBLICATIONS OF THE LAND LOCOMOTION LABORATORY, RESEARCH DIVISION, OTAC DETROIT ARSENAL, CENTER LINE, MICHIGAN

A. REPORTS

NO.	TITLE
1	Minutes of the First Meeting of the Scientific Advisory Committee (Tech Memo M-01)
2	Preliminary Study of Snow Values Related to Vehicle Performance (Tech Memo M-02)
3	An Investigation of Spades for Recovery Vehicles (Tech Memo M-03)
4	Techniques for the Evaluation of Track and Road-Wheel Design (Tech Memo M-04)
13	Terrain Evaluation in Automotive Off-the-Road Operation
14	Application of a Variable Pitch Propeller as a Booster of Lift and Thrust for Amphibian Vehicles
15	Mobility on Land; Challenge and Invitation
16	Minutes of the Second Meeting of the Scientific Advisory Committee
18	An Analysis of New Techniques for the Estimation of Footing Sinkage in Soils
19	An Investigation of Gun Anchoring Spades Under the Action of Inpact Loads
20	Artificial Soils for Laboratory Studies in Land Locomotion
22	An Introduction to Research on Vehicle Mobility
23	Study of Snow Values Related to Vehicle Performance
25	Drag Coefficients in Locomotion over Viscous Soils
26	Evaluation of Tires for the XM410 8x8, 2-1/2 Ton Truck
28	Effect of Impenetrable Obstacles on Vehicle Operational Speed

29 Obstacle Performance of Wheeled Vehicles

31 Performance and Design of Crawler Tractors

32 A New Booster of Lift and Thrust for Amphibian Vehicles

33 Determination of Soil Sinkage Parameters by Means of Rigid Wheels, Part I

35 Estimation of Sinkage in Off-the-Road Locomotion

40 Operational Definition of Mechanical Mobility of Motor Vehicles

41 A Definition of Soil Trafficability

43 Study on Cross Country Locomotion

46 Prediction of "WES Cone Index" by Means of a Stress-Strain Function of Soils

48 Behavior of a Linear One Degree of Freedom Vehicle Moving with Constant Velocity on a Stationary Gaussian Random Track

54 Drag Coefficients of Locomotion over Viscous Soils, Part II

55 Operational Definition of Mechanical Mobility

56 On the Behavior of a Linear Two Degree of Freedom Vehicle Moving with Constant Velocity on a Track Whose Contour is a Stationary Random Process

57 Determination of k_c, k_ϕ and n Values by Means of Circular Plates, Modified Procedure

58 The Turning Behavior of Articulated Track Laying Vehicles

59 Mobility Studies

60 Evaluation of Condual Tire Model

61 A Simplified Method for the Determination of Bulldozing Resistance

62 Analysis on Towed Pneumatic Tire Moving on Soft Ground

63 The Mechanics of the Triaxial Tests for Soils

64 Triaxial Tests on Saturated Sand and on Sands Containing Some Clay

65 On the Statistical Analysis of the Motion of Some Simple Vehicles Moving on a Random Track

66 On the Statistical Analysis of the Motion of Some Simple Two Dimensional Linear Vehicles Moving on a Random Track

67 Effects of Sinkage Speed on Land Locomotion Soil Values

68 Over-Snow Vehicles Performance Studies

69 An Analysis of the Drawbar-Pull vs Slip Relationship for Track Laying Vehicles

70 Stresses and Deformation under Oblique Loads

71 Mechanics of Walking Vehicles

72 On the Statistical Properties of the Ground Contour and its Relation to the Study of Land Locomotion

B. GENERAL PUBLICATIONS

a. Research Report No. 1

b. Research Report No. 2

c. Research Report No. 3

d. Research Report No. 4

e. Research Report No. 5

DISTRIBUTION LIST

Commanding General
Aberdeen Proving Gd, Md
ATTN: Tech Library 4

Commandant
Ordnance School
Aberdeen Proving Gd, Md 1

British Joint Service Mission
Ministry of Supply
P. O. Box 680
Benjamin Franklin Station
Washington, D. C.
ATTN: Reports Officer 2

Canadian Army Staff
2450 Massachusetts Avenue
Washington, D. C. 4

British Joint Service Mission
Ministry of Supply Staff
1800 K Street, N. W.
Washington, D. C. 6

Director
Waterways Experiment Station
Vicksburg, Mississippi 3

Unit X
Documents Expediting Project
Library of Congress
Washington, D. C.
Stop 303 4

Exchange and Gift Div
Library of Congress
Washington 25, D. C. 1

United States Navy
Industrial College of the
Armed Forces
Washington, D. C.
ATTN: Vice Deputy
Commandant 1

Dept of National Defense
Dr. N. W. Morton
Scientific Advisor
Chief of General Staff
Army Headquarters
Ottawa, Ontario, Canada 1

Chief of Ordnance
Department of the Army
Washington 25, D. C.
ATTN: ORDTM 2

Commanding Officer
Office of Ordnance Research
Box CM, Duke Station
Durham, North Carolina 3

Chief
Office of Naval Research
Washington, D. C. 1

Headquarters
Ordnance Weapons Command
Research & Development Div
Rock Island, Illinois
ATTN: ORDOW-TB 2

Commanding Officer
Diamond Ordnance Fuze Lab
Washington 25, D. C.
ATTN: ORDTL 012 2

DISTRIBUTION LIST (Cont'd)

Detroit Arsenal AFF Liaison Office, CONARC	10
Detroit Arsenal Canadian Liaison Office	4
Detroit Arsenal Technical Library	2
Superintendent U. S. Military Academy West Point, New York ATTN: Prof. Of Ordnance	1
Superintendent U. S. Naval Academy Annapolis, Md	1

ABSTRACT

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY Army Tank Automotive Center Warren, Michigan	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP (For ASTIA use only)
3. REPORT TITLE On the Statistical Analysis of Linear Vehicle Dynamics	
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Formal	
5. AUTHOR(S) (Last name, first name, initial) Bogdanoff, J. L. and Kozin, F.	
6. PUBLICATION DATE March 1962	7. TOTAL NO OF PAGES 52 p.
8. ORIGINATOR'S REPORT NO(S) 7824 LL 73	9a. CONTRACT OR GRANT NO b. PROJECT NO 5510.11.822 c. TASK NO
10. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) D/A Project: 570-05-001	11. SUPPLEMENTARY NOTES (For ASTIA use only)
12. RELEASE STATEMENTS (For ASTIA use only)	

13 AUTHORS' KEY TERMS - UNCLASSIFIED ONLY

1. Statistical analysis	7. Frame acceleration	13. Vehicles
2. Linear vehicle dynamics	8. Ride roughness	14.
3. Wheel base length	9. Damping	15.
4. Idealized tire imprint	10. Parameters	16.
5. Constant velocity	11. Speed	17.
6. Spectral density	12. Land locomotion	18.

14. ASTIA DESCRIPTORS (For ASTIA use only)

15. IDENTIFIERS - UNCLASSIFIED ONLY (e.g., Model numbers; weapon system, project, chemical compound and trade names)

OTA FORM
24 APR 64 5500 (Onetime)

16. BODY OF ABSTRACT (Including if applicable, Purpose; Method or approach; Results; Conclusions, applications and recommendations.)

Statistical analyses of the dynamics of some two-dimensional linear vehicles traveling on a rough track are performed to determine the influence on two aspects of vehicle ride of a set of parameters which include wheel base length, idealized tire imprint length, speed, and damping constant. It is assumed that the vehicles move with constant horizontal velocity on a second order, weakly stationary and mean square continuous random track with contact maintained at all times between the idealized tires and the track. The two aspects of vehicle ride used as measures of the ride roughness are peak value of power spectral density and variance of frame acceleration, the frame acceleration being either vertical at the c.g. of frame, vertical at the point over idealized wheel, or angular (pitching).

For the same speed, damping, power spectral density for the track, and two particular vehicles, the idealized tire imprint length was a relatively unimportant parameter over a fairly large range of values. On the other hand, one parameter which included the wheel base length was found to be important under the same conditions.

Four sets of parameter values were found which at the same speed produced best or optimal rides for vertical acceleration at the frame c. g. and over the wheel, depending upon which measure of ride roughness was employed. The influence of speed was then examined on vehicles having these sets of parameter values. In all cases, increasing speeds produced sharp increases in ride roughness.

17. INDEXING ANNOTATION

Statistical Analysis of Linear Vehicle Dynamics